

One-loop test of free $SU(N)$ adjoint model holography

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ABSTRACT: We consider the holographic duality where the CFT side is given by $SU(N)$ adjoint free scalar field theory. Compared to the vector models, the set of single trace operators is immensely extended so that the corresponding AdS theory also contains infinitely many massive higher spin fields on top of the massless ones. We compute the one-loop vacuum energy of these AdS fields to test this duality at the subleading order in large N expansion. The determination of the bulk vacuum energy requires a proper scheme to sum up the infinitely many contributions. For that, we develop a new method and apply it first to calculate the vacuum energies for the first few ‘Regge trajectories’ in AdS_4 and AdS_5 . In considering the full vacuum energy of AdS theory dual to a matrix model CFT, we find that there exist more than one available prescriptions for the one-loop vacuum energy. Taking a particular prescription, we determine the full vacuum energy of the AdS_5 theory, whereas the AdS_4 calculation still remains technically prohibitive. This result shows that the full vacuum energy of the AdS_5 theory coincides with minus of the free energy of a single scalar field on the boundary. This is analogous to the $O(N)$ vector model case, hence suggests an interpretation of the *positive* shift of the bulk coupling constant, i.e. from $N^2 - 1$ to N^2 .

KEYWORDS: $1/N$ Expansion, AdS-CFT Correspondence, Higher Spin Gravity

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1 Introduction

Recently, there has been significant progress in the vectorial AdS/CFT correspondence [1, 2]. It relates free/critical CFTs in the $O(N)$ (or $U(N)$) vector multiplet to the Vasiliev’s higher spin theory [3, 4] with a certain boundary conditions. An important feature of this duality is the precise one-to-one correspondence between the spectrum of ‘light’ conformal primaries¹ on the CFT side and the spectrum of quadratic fluctuations near the AdS vacuum. In fact, the match of the spectrum — shown by Flato and Fronsdal [5] — predates the AdS/CFT conjecture and even Vasiliev’s theory. It states that the tensor product of two singleton representations of $so(2, 3)$ — the scalar one, Rac or the spinor one, Di — can be decomposed into the infinite sum of massless higher spin representations:

$$\text{Rac} \otimes \text{Rac} = \bigoplus_{s=0}^{\infty} \mathcal{D}(s+1, s), \quad \text{Di} \otimes \text{Di} = \mathcal{D}(2, 0) \oplus \bigoplus_{s=1}^{\infty} \mathcal{D}(s+1, s), \quad (1.1)$$

where $\mathcal{D}(\Delta, s)$ is the representation with spin s and the conformal dimension Δ . This mathematical theorem can be translated into the AdS/CFT language as: all bilinear scalar operators (the tensor product of Rac) in free conformal scalar/spinor fields in three dimensions have an one-to-one correspondence with the massless higher spin fields in the bulk of AdS₄ (the representations of $\mathcal{D}(s+1, s)$). The fact that the CFT fields are in the vectorial representation — not in the adjoint one — singles out only bilinear operators as the single trace operators. The contributions of higher trace operators are suppressed in the large N limit, hence we are left with a minimal set of operators in the spectrum.

A trivial but important property of this duality is that the boundary CFTs do not have any $1/N$ subleading contributions as they are free theories. An immediate implication of this property towards the bulk physics is the absence of any quantum corrections. This is a remarkable feature because it necessitates a precise cancellation of infinitely many loop diagrams in the bulk. This aspect has been examined in the series of the papers [6–8] and [9–13]² where the authors considered the simplest example, the vanishment of the one-loop vacuum energy.³ Since the vacuum energy in AdS _{$d+1$} ought to be dual to the CFT zero-point function, they can only depend on the size of the radius of the boundary S^d .

¹By ‘light’ we mean that the conformal dimension of the primary does not scale with N .

²See also [14–17] for related discussions on the conformal higher spin theory.

³By ‘vacuum energy’ we mean the log of the partition function about Euclidean AdS _{$d+1$} . We note that this terminology is different from that adopted in [8], where the same object was referred to as ‘ S^d partition function’.

The summation of the vacuum energies over all field contents results in an infinite series:

$$\sum_{s=0}^{\infty} \left(\text{circle with } s \text{ inside} \right) = \sum_{s=0}^{\infty} \Gamma_s^{(1)}(z) = \Gamma^{(1)}(z), \quad (1.2)$$

where $\Gamma_s^{(1)}(z)$ is the UV regularized (with a regulator z) vacuum energy with the massless spin s field in the loop. Two different methods have been considered to analyze this series. In the first method, the summation over s is carried out before getting the function $\Gamma^{(1)}(z)$. The resulting vacuum energy is free from UV divergence and vanishes as $z \rightarrow 0$, for the minimal Vasiliev theory. This method does not require an additional regularization scheme and it is used for even d cases where $\Gamma_s^{(1)}(z)$ have relatively simple forms. In odd d , however, the expression of $\Gamma_s^{(1)}(z)$ is more involved such that we cannot identify $\Gamma^{(1)}(z)$ with preceding method. This necessitates another approach. In the second method, we take only the finite part $\Gamma_s^{(1)\text{ren}}$ of $\Gamma_s^{(1)}(z)$ (and neglect the divergent part) to end up with the series $\sum_{s=0}^{\infty} \Gamma_s^{(1)\text{ren}}$. This series is divergent so requires a new regularization in order to show that it indeed vanishes.

Motivated by these developments, we study the quantum property of the AdS theory dual to the free scalar CFT in $SU(N)$ adjoint representation [18],

$$S_{\text{CFT}}[\phi] = \int d^d x \text{Tr} \left[\phi^\dagger \square \phi \right], \quad (1.3)$$

by focusing on its vacuum energy. There are several reasons which lead us to do so. Firstly, in contrast to the vector models, the set of single trace operators includes not only bilinear but also operators multi-linear in the field ϕ . This greatly extends the field content of the dual theory as compared to Vasiliev's theory. Standard AdS/CFT considerations lead us to expect that the holographic dual of such theory is the Vasiliev higher spin theory coupled to infinitely many massive higher spin fields. The whole spectrum organizes itself into infinitely many 'Regge trajectories',⁴ each of which forms a 'matter' multiplet of the higher spin algebra.⁵ Secondly, the putative holographic duality would closely mimic the dualities involving string theory in AdS in many ways. In particular, the theory already has interesting thermodynamics, exhibiting a Hagedorn phase transition [20, 21] much like string theory [22]. Thirdly, as a consequence of the usual AdS/CFT dictionary, we expect that the free CFT limit of stringy AdS/CFT dualities corresponds to taking the tensionless limit of string theory in AdS [23–35].

Further, since we are working with a free CFT, the determination of the spectrum in closed form is available, even if it is a complicated task. As in the Flato-Fronsdal theorem which dictates the spectrum of the vector models, the spectrum of $SU(N)$ adjoint model can be identified by decomposing the multiple tensor products of singletons into irreducible

⁴By Regge trajectory, we mean the set of AdS fields dual to the CFT operators made by a fixed number of conformal fields: the n -th trajectory is dual to the CFT operators involving $n + 1$ conformal fields.

⁵See appendix C of [19] for a recent discussion.

representations:

$$\bigoplus_{n=2}^{\infty} \text{Rac}^{\otimes n} = \bigoplus_{\Delta,s} N_{\Delta,s} \mathcal{D}(\Delta, s), \quad (1.4)$$

where $N_{\Delta,s}$ is the multiplicity of the representation $\mathcal{D}(\Delta, s)$ and n is the number of conformal fields. While doing this decomposition, the tensor product should be properly projected for the consistency with the cyclic invariance of trace [36–41].

We now present a brief overview of our strategy. Firstly, we determine the operator spectra of the CFT which will be identified with the spectrum of the bulk theory. To determine the spectrum, we mainly take the most standard way of character analysis [42, 43] but will also present the oscillator analysis for the $d = 3$ case. These analyses give us information about the operator spectrum so that we can calculate in principle the corresponding AdS vacuum energies, knowing the one-loop contribution from each bulk field [44–48]. However the difficulties arise in the summation of the vacuum energies from different fields. This is due to both the increasing complexity of the spectrum as higher and higher Regge trajectories are included, as well as the careful regularization of many formally divergent sums, as was already encountered in the vector model case [6–8]. In order to surpass this problem, we introduce a new technique which enables us to access the resummed vacuum energy directly from the character bypassing the steps of decomposition and resummation. This is realized in terms of a functional \mathcal{F} whose input is the character χ (or generalized partition function⁶) of the CFT and the output is the UV regularized AdS vacuum energy:

$$\sum_{\Delta,s} N_{\Delta,s} \left(\bigoplus_{\Delta,s} \right) = \mathcal{F}[\chi](z). \quad (1.5)$$

In this paper, we revisit the Vasiliev’s theories as test examples of applying the new method. Then, we challenge the AdS theory dual to the $\text{SU}(N)$ adjoint matrix model.

Organization of the paper. The paper is organized as follows. We begin with a review of unitary irreducible representations (UIRs) of the d -dimensional conformal algebra $so(2, d)$ in section 2, including the introduction of singleton representations Di and Rac , construction of (reducible) representations by taking tensor products of singletons, as well as character formulae for the various UIRs. Based on the character of conformal algebra, we present decomposition rules of singleton tensor products, by using a generating function method. We also discuss an oscillator construction for arriving at these decomposition rules. Section 3 contains a review of the heat kernel and zeta function formalism for computing one-loop effects in Euclidean Anti-de Sitter space, and general expectations from AdS/CFT duality for matching with free CFT answers. In section 4, we compute the spectral zeta function for AdS_4 using the results for the spectrum found in section 2,

⁶By this we mean ‘refined’ partition function $\text{Tr} \left(e^{-\beta H + \sum_i \alpha_i J^i} \right)$ computed over the the *one-particle* Hilbert space of the theory. Here β is the temperature and α_i are chemical potentials for the Cartan subalgebra of $so(d) \subset so(2, d)$.

and also by the new formalism alluded to above. Section 5 contains the extension of the above results to AdS_5 where we also discuss how mixed symmetry fields may be taken into account. Section 6 summarizes and concludes this paper and discusses related issues. Appendices contain various additional details.

2 Operator spectrum of free $\text{SU}(N)$ adjoint model

In this section, we will study the operator spectrum of free matrix models. Many works have been devoted to this task [36–43]. Putting aside other interesting models, we only consider the simplest case of free scalar $\text{SU}(N)$ adjoint model. According to the standard scheme of AdS/CFT correspondence, in the large N limit, single trace operators are dual to the *single-particle* states (or fields) in the bulk theory. Any single trace operator in a scalar CFT can be written as a linear combination of the operators,

$$\text{Tr} \left[(\partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_l} \phi) (\partial_{\nu_1} \partial_{\nu_2} \cdots \partial_{\nu_m} \phi) \cdots (\partial_{\rho_1} \partial_{\rho_2} \cdots \partial_{\rho_n} \phi) \right]. \quad (2.1)$$

These operators are in general reducible with respect to the conformal symmetry, and can be decomposed into unitary irreducible representations (UIR). Since the d -dimensional conformal symmetry and the isometry of AdS_{d+1} are equivalent — they are both $so(2, d)$ — the single trace operators carrying UIRs of the conformal symmetry are in one-to-one correspondence with the bulk fields carrying UIRs of the AdS isometry. The decomposition does not mix the operators (2.1) with different number of scalar fields in the trace, hence can be performed for a given number of scalar fields.

The identification of all single trace operators reduces to the decomposition of the operators (2.1) into UIRs for any number of scalar fields in the large N limit. Exact expressions of the decompositions — that is, the expression of UIR single trace operators in terms of ϕ — involve rather complicated (anti)symmetrizations and contractions of indices. However, for our purposes, it is sufficient to find out the UIR labels of the resulting operators from the decompositions. The latter task can be conveniently carried out relying on the representation theory of conformal symmetry $so(2, d)$. The free scalar field carries a short UIR, called scalar singleton, hence we are to analyze the decomposition of multiple tensor products of singletons into UIRs of $so(2, d)$. The Lie algebra character is one of the most convenient tools for this analysis. In the following, we begin with a brief summary of the UIRs and the characters of $so(2, d)$. More detailed accounts and derivations can be found in [49].⁷

2.1 Review: UIRs and characters of $so(2, d)$

The conformal algebra in d -dimension is isomorphic to $so(2, d)$. The latter is generated by M_{AB} , whose commutation relations are given by

$$[M_{AB}, M_{CD}] = i (\eta_{A[C} M_{B]D} - \eta_{B[C} M_{A]D}), \quad (2.2)$$

⁷A particularly accessible ‘physicist’s account’ for the UIRs of $so(2, d)$ is available in [50]. The UIRs of $so(2, 3)$ and $so(2, 4)$ were first constructed in [51] and [52] respectively. For the symmetries of the singleton representation, see [53, 54].

where the indices A, B, \dots run over $+, -, 1, 2, \dots, d$ while $a, b, \dots = 1, 2, \dots, d$. The non-vanishing components of the metric are $\eta_{\pm\mp} = 1$ and $\eta_{ab} = \delta_{ab}$. Its lowest weight (LW) representations $\mathcal{V}(\Delta, \ell)$ are labeled by those of the $so(2) \oplus so(d)$ maximally compact subalgebra generated by $E = M_{+-}$ and M_{ab} . The Δ corresponds to the eigenvalue of the generator E , and the $\ell = (\ell_1, \dots, \ell_h)$ labels the irreducible representation of $so(d)$ with

$$\ell_1 \geq \dots \geq \ell_{h-1} \geq |\ell_h|; \quad h = \left\lceil \frac{d}{2} \right\rceil; \quad \ell_h \geq 0 \quad \text{if} \quad d = 2h + 1, \quad (2.3)$$

where ℓ_i 's are either all integers, or all half-integers for a given representation.

The character of $\mathcal{V}(\Delta, \ell)$ is given by

$$\chi_{\Delta, \ell}(q, \mathbf{x}) = \text{Tr}_{\mathcal{V}(\Delta, \ell)} \left[q^E x_1^{M_{12}} \dots x_h^{M_{h(h+1)}} \right] = q^\Delta P_d(q, \mathbf{x}) \chi_\ell^{so(d)}(\mathbf{x}), \quad (2.4)$$

where $\mathbf{x} = (x_1, \dots, x_h)$ and $P_d(q, \mathbf{x})$ is,

$$P_d(q, \mathbf{x}) = \frac{1}{(1-q)^{d-2h}} \prod_{i=1}^h \frac{1}{(1-qx_i)(1-qx_i^{-1})}, \quad (2.5)$$

while $\chi_\ell^{so(d)}(\mathbf{x})$ is the character of ℓ representation of $so(d)$. For even d , the character has a form

$$\chi_\ell^{so(2h)}(\mathbf{x}) = \frac{\det [x_i^{k_j} + x_i^{-k_j}] + \det [x_i^{k_j} - x_i^{-k_j}]}{2 \Delta(\mathbf{x})}, \quad (2.6)$$

and for odd d ,

$$\chi_\ell^{so(2h+1)}(\mathbf{x}) = \frac{\det [x_i^{k_j} - x_i^{-k_j}]}{\Delta(\mathbf{x}) \prod_{i=1}^h \left(x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}} \right)}, \quad (2.7)$$

with $k_i = \ell_i + \frac{d}{2} - i$. The $\Delta(\mathbf{x})$ is the Vandermonde determinant,

$$\Delta(\mathbf{x}) = \prod_{1 \leq i < j \leq h} \left(x_j + x_j^{-1} - x_i - x_i^{-1} \right). \quad (2.8)$$

The irreducible representation $\mathcal{D}(\Delta, \ell)$ of $so(2, d)$ is the quotient of $\mathcal{V}(\Delta, \ell)$ by its maximal $so(2, d)$ -invariant subspace $\mathcal{D}(\Delta', \ell')$ with a certain Δ' and ℓ' . The corresponding character of $\mathcal{D}(\Delta, \ell)$ is then given by

$$\chi_{\mathcal{D}(\Delta, \ell)}(q, \mathbf{x}) = \chi_{\Delta, \ell}(q, \mathbf{x}) - \chi_{\mathcal{D}(\Delta', \ell')}(q, \mathbf{x}). \quad (2.9)$$

In the following, we summarize the UIRs $\mathcal{D}(\Delta, \ell)$ of $so(2, d)$ and their characters $\chi_{\mathcal{D}(\Delta, \ell)}$. For that, we need to define the number p as

$$\ell_1 = \ell_2 = \dots = |\ell_p| > \ell_{p+1}, \quad (2.10)$$

for a $so(d)$ representation $\ell = (\ell_1, \dots, \ell_h)$.

Long representations. We consider first the representations $\mathcal{D}(\Delta, \ell)$ whose $so(d)$ part, ℓ , satisfies

$$1 \leq p < \frac{d}{2}, \quad \ell_1 \geq d - 2h, \quad (2.11)$$

which excludes the scalar $\ell = (0, \dots, 0)$ and the spinor $\ell = (\frac{1}{2}, \dots, \pm\frac{1}{2})$ representations in odd dimensions and all the representations with $p = h$ in even d . For the above class of representations, the unitarity bound is given in [55] by

$$\Delta_\ell = \ell_1 + d - p - 1. \quad (2.12)$$

Above the bound, the LW representation does not develop any invariant subspace, hence

$$\mathcal{D}(\Delta, \ell) = \mathcal{V}(\Delta, \ell) \quad [\Delta > \Delta_\ell]. \quad (2.13)$$

These are *long* representations. Below the bound, $\Delta < \Delta_\ell$, the representation becomes non-unitary.

Semi-short representations. If Δ saturates the unitarity bound, i.e. $\Delta = \Delta_\ell$, the representation gets shortened as

$$\mathcal{D}(\Delta_\ell, \ell) = \mathcal{V}(\Delta_\ell, \ell) / \mathcal{D}(\Delta_\ell + 1, \bar{\ell}), \quad (2.14)$$

where $\bar{\ell} = (\bar{\ell}_1, \dots, \bar{\ell}_h)$ is defined through ℓ as

$$\bar{\ell}_i = \ell_i - \delta_{pi}. \quad (2.15)$$

These representations are often referred as to *semi-short* for the distinction from shorter representations, namely singletons, which will be referred as to *short*. Notice that for $p > 1$, we have $\Delta_\ell + 1 = \Delta_{\bar{\ell}}$, hence the invariant subspace corresponds again to a semi-short representation $\mathcal{D}(\Delta_{\bar{\ell}}, \bar{\ell})$. Therefore, the quotient process should be repeated recursively until one reaches $p = 1$.⁸ The characters of semi-short representations are given by

$$\chi_{\mathcal{D}(\Delta_\ell, \ell)}(q, \mathbf{x}) = \chi_{\Delta_\ell, \ell}(q, \mathbf{x}) - \chi_{\mathcal{D}(\Delta_\ell + 1, \bar{\ell})}(q, \mathbf{x}), \quad (2.16)$$

which again can be recursively decomposed into $\chi_{\Delta_\ell, \ell}$ with different ℓ 's. Both the semi-short and long representations have the Gelfand-Kirillov (GK) dimension d [56]: they can be described by a function of d continuous variables. Hence they can be realized either as an operator on d -dimensional boundary without an on-shell condition or as a field living on AdS_{d+1} . More precisely, in the bosonic case, $\mathcal{D}(\Delta, \ell)$ can be realized as a boundary irreducible $so(d)$ -tensor operator,⁹

$$\mathcal{O}_\Delta^{a_1(\ell_1), \dots, a_h(\ell_h)}(x), \quad (2.17)$$

or as a bulk irreducible $so(1, d)$ -tensor field,

$$\varphi_{\mu_1(\ell_1), \dots, \mu_h(\ell_h)}(z, x), \quad (2.18)$$

⁸This has been shown explicitly by means of an oscillator construction in [42] for the case of $d = 4$.

⁹Here, $a_i(\ell_i)$ is a shorthand notation for a group of fully symmetric ℓ_i indices $a_{i|1} a_{i|2} \cdots a_{i|\ell_i}$.

with the mass squared [57],

$$M_{\Delta,\ell}^2 = \frac{\Delta(\Delta - d) - \sum_{i=1}^h \ell_i}{R^2}, \quad (2.19)$$

where R is the radius of AdS. At the shortening point $\Delta = \Delta_\ell$, the boundary operator satisfies a conservation condition:

$$Y_{\bar{\ell}} \left[\partial_{a_p | \ell_p} \mathcal{O}_{\Delta_\ell}^{a_1(\ell_1), \dots, a_h(\ell_h)}(x) \right] = 0, \quad (2.20)$$

whereas the bulk field admits a gauge symmetry

$$\delta \varphi_{\mu_1(\ell_1), \dots, \mu_h(\ell_h)}(z, x) = Y_\ell \left[\partial_{\mu_p | \ell_p} \varepsilon_{\mu_1(\bar{\ell}_1), \dots, \mu_h(\bar{\ell}_h)}(z, x) \right]. \quad (2.21)$$

Here Y_ℓ is the projection operator to the irreducible Young diagram ℓ (see [55, 58, 59] for the details). Notice that the conservation condition corresponds to the invariant subspace $\mathcal{D}(\Delta_{\bar{\ell}}, \bar{\ell})$.

Before moving to the short representation, let us consider the example of the symmetric tensor representation $\ell = (\ell, 0, \dots, 0) =: (\ell, \mathbf{0})$, whose unitary bound is given by $\Delta_{(\ell, \mathbf{0})} = \ell + d - 2$. The character is given simply by

$$\chi_{\mathcal{D}(\Delta_{(\ell, \mathbf{0})}, (\ell, \mathbf{0}))}(q, \mathbf{x}) = \chi_{\Delta_{(\ell, \mathbf{0})}, (\ell, \mathbf{0})}(q, \mathbf{x}) - \chi_{\Delta_{(\ell, \mathbf{0})}+1, (\ell-1, \mathbf{0})}(q, \mathbf{x}), \quad (2.22)$$

since $p = 1$ in this case. This representation can be realized either as a conserved current $\mathcal{O}^{a_1 \dots a_\ell}$ on the boundary or as a symmetric gauge field $\varphi_{\mu_1 \dots \mu_\ell}$ in the bulk.

Short representations: singletons. The condition (2.11) leaves three exceptional cases, where we get short representations, instead of semi-short ones, when Δ is on the boundary of unitarity. The short representations have one less GK dimension, that is $d-1$, hence do not admit a standard field theoretic realization in the $d+1$ dimensional bulk (see however the attempts [60]). More suitable realization of them is as boundary conformal field operator subject to certain on-shell conditions.

Scalar singleton. The first case is the scalar representation, $\ell = (0, \dots, 0) =: \mathbf{0}$, where the unitarity bound reads

$$\Delta_0 = \frac{d-2}{2}. \quad (2.23)$$

Above the bound $\Delta > \Delta_0$, we get a long representation. On the border of the unitarity, $\Delta = \Delta_0$, we have the scalar singleton,

$$\mathcal{D}(\Delta_0, \mathbf{0}) = \mathcal{V}(\Delta_0, \mathbf{0}) / \mathcal{V}(\Delta_0 + 2, \mathbf{0}). \quad (2.24)$$

Its character is given by

$$\chi_{\mathcal{D}(\Delta_0, \mathbf{0})}(q, \mathbf{x}) = \chi_{\Delta_0, \mathbf{0}}(q, \mathbf{x}) - \chi_{\Delta_0+2, \mathbf{0}}(q, \mathbf{x}) = q^{\frac{d-2}{2}} (1 - q^2) P_d(q, \mathbf{x}). \quad (2.25)$$

The scalar singleton representation can be realized as a conformal scalar ϕ on the boundary and the subspace $\mathcal{V}(\Delta_0 + 2, \mathbf{0})$ corresponds to the l.h.s. of the equation of motion for the conformal scalar, $(\square + \frac{d-2}{4(d-1)} \mathcal{R})\phi = 0$.

Spinor singleton. The second case is the spinor representation $\ell = (\frac{1}{2}, \dots, \frac{1}{2}) =: \frac{1}{2}$, where the unitarity requires

$$\Delta \geq \Delta_{\frac{1}{2}} = \frac{d-1}{2}. \quad (2.26)$$

On the border, we get the spinor singleton,

$$\mathcal{D}\left(\Delta_{\frac{1}{2}}, \frac{1}{2}\right) = \mathcal{V}\left(\Delta_{\frac{1}{2}}, \frac{1}{2}\right) / \mathcal{V}\left(\Delta_{\frac{1}{2}} + 1, \frac{1}{2}\right). \quad (2.27)$$

It can be realized as a Dirac spinor ψ on the boundary, and the subspace $\mathcal{V}(\Delta_{\frac{1}{2}} + 1, \frac{1}{2})$ corresponds to the equation of motion $\not{\partial}\psi = 0$. The character is given by

$$\chi_{\mathcal{D}(\Delta_{\frac{1}{2}}, \frac{1}{2})}(q, \mathbf{x}) = \chi_{\Delta_{\frac{1}{2}}, \frac{1}{2}}(q, \mathbf{x}) - \chi_{\Delta_{\frac{1}{2}}+1, \frac{1}{2}}(q, \mathbf{x}) = q^{\frac{d-1}{2}}(1-q) \chi_{\frac{1}{2}}^{so(d)}(\mathbf{x}) P_d(q, \mathbf{x}),$$

with the $so(d)$ one,

$$\chi_{\frac{1}{2}}^{so(d)}(\mathbf{x}) = \prod_{i=1}^h \left(x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}} \right). \quad (2.28)$$

Higher spin singleton. When d is even, the representations $\mathcal{D}(\Delta, \ell)$ with $\ell = \mathbf{s}_{\pm} := (s, \dots, s, \pm s)$ also develop short representations on the unitarity bound:

$$\Delta_{\mathbf{s}_{\pm}} = s + \frac{d-2}{2}. \quad (2.29)$$

Since the above can be also written as $\Delta_{\mathbf{s}_{\pm}} = s + h - 1$, it corresponds in fact to particular cases of (2.12). Moreover, for $\ell = \mathbf{0}$ and $\frac{1}{2}_{+}$, it coincides with the bounds (2.23) and (2.26) for the scalar and spinor singletons. Again, above the bound the representations are long, while on the border, we get the short representations,

$$\mathcal{D}(\Delta_{\mathbf{s}_{\pm}}, \mathbf{s}_{\pm}) = \mathcal{V}(\Delta_{\mathbf{s}_{\pm}}, \mathbf{s}_{\pm}) / \mathcal{D}(\Delta_{\mathbf{s}_{\pm}} + 1, \bar{\mathbf{s}}_{\pm}), \quad (2.30)$$

where $\bar{\mathbf{s}}_{\pm}$ is defined in (2.15) with $(\bar{s}_{-})_h = -(s-1)$. For a more explicit expression of character, we define

$$\mathbf{e}_{\pm, n} = \underbrace{(0, \dots, 0)_{h-n}}_{h-n}, \underbrace{(1, \dots, 1, \pm 1)_n}_n, \quad (2.31)$$

then we get

$$\chi_{\mathcal{D}(\Delta_{\mathbf{s}_{\pm}}, \mathbf{s}_{\pm})}(q, \mathbf{x}) = \sum_{n=0}^h (-1)^n \chi_{\Delta_{\mathbf{s}_{\pm}}+n, \mathbf{s}_{\pm}-\mathbf{e}_{\pm, n}}(q, \mathbf{x}). \quad (2.32)$$

In the integer s cases, higher spin singletons can be realized by boundary tensor fields $\varphi^{a_1(s), \dots, a_h(s)}$ and the subspace $\mathcal{D}(\Delta_{\mathbf{s}_{\pm}} + 1, \bar{\mathbf{s}}_{\pm})$ corresponds to the conservation condition (2.20). In terms of dual fields,¹⁰

$$\pi_{b_1[h], \dots, b_s[h]} = \epsilon_{b_1[h] a_{1|1} a_{2|1} \dots a_{h|1}} \dots \epsilon_{b_s[h] a_{1|s} a_{2|s} \dots a_{h|s}} \varphi^{a_1(s), \dots, a_h(s)}, \quad (2.33)$$

¹⁰Here, $b_i[h]$ is a shorthand notation for a group of fully anti-symmetric h indices $b_{i|1} b_{i|2} \dots b_{i|h}$.

the conservation condition get simplified as

$$\partial_{b_1} \pi_{b_1[h], \dots, b_s[h]} = 0. \quad (2.34)$$

This is the Bargmann-Wigner equation for massless higher spin field [61]. The \pm of \mathbf{s}_\pm defines different parity decompositions depending on the dimensionality. For even h (that is $d = 4m$), they correspond to self-dual and anti-self-dual field, whereas for odd h (that is $d = 4m + 2$), they correspond to chiral and anti-chiral field.

2.1.1 Examples of lower dimensions

Let us conclude the review of the UIRs and characters of $so(2, d)$ with the examples of lower dimensions $d = 2, 3, 4$.

$so(2, 2)$. From (2.4), the character of long representations are

$$\chi_{\Delta, \ell}(q, x) = q^\Delta P_2(q, x) \chi_\ell^{so(2)}(x), \quad (2.35)$$

with P_2 and $so(2)$ character $\chi^{so(2)}$ given by

$$P_2(q, x) = \frac{1}{(1 - qx)(1 - qx^{-1})}, \quad \chi_\ell^{so(2)}(x) = x^\ell. \quad (2.36)$$

Since $so(2, 2) \simeq so(1, 2) \oplus so(1, 2)$, the above character can be decomposed into that of $so(1, 2)$ as

$$\chi_{\Delta, \ell}(q, x) = \chi_j^{so(1, 2)}(z_+) \chi_{\bar{j}}^{so(1, 2)}(z_-), \quad (2.37)$$

where Δ, ℓ are related to j, \bar{j} as

$$\Delta = j + \bar{j}, \quad \ell = j - \bar{j}, \quad qx = z_+, \quad qx^{-1} = z_-, \quad (2.38)$$

and the character for $so(1, 2)$

$$\chi_j^{so(1, 2)}(z) = \frac{z^j}{1 - z}. \quad (2.39)$$

Here, short representations correspond to the holomorphic or anti-holomorphic ones,

$$\chi_{\mathcal{D}(s, \pm s)}(q, x) = \chi_{s, \pm s}(q, x) - \chi_{s+1, \pm(s-1)}(q, x) = \chi_s^{so(1, 2)}(z_\pm). \quad (2.40)$$

$so(2, 3)$. In $d = 3$, the character for long representations is given by

$$\chi_{\Delta, \ell}(q, x) = q^\Delta P_3(q, x) \chi_\ell^{so(3)}(x), \quad (2.41)$$

with P_3 and $so(3)$ character given by

$$P_3(q, x) = \frac{1}{(1 - q)(1 - qx)(1 - qx^{-1})}, \quad \chi_\ell^{so(3)}(x) = \frac{x^{\ell + \frac{1}{2}} - x^{-\ell - \frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}. \quad (2.42)$$

The only semi-short representation is the symmetric tensor one,

$$\chi_{\mathcal{D}(\ell+1, \ell)}(q, x) = q^{\ell+1} P_3(q, x) \left(\chi_\ell^{so(3)}(x) - q \chi_{\ell-1}^{so(3)}(x) \right). \quad (2.43)$$

For short representations, we have scalar and spinor representations corresponding to the characters,

$$\chi_{\text{Rac}}(q, x) = \frac{q^{\frac{1}{2}}(1 + q)}{(1 - qx)(1 - qx^{-1})}, \quad \chi_{\text{Di}}(q, x) = \frac{q(x^{\frac{1}{2}} + x^{-\frac{1}{2}})}{(1 - qx)(1 - qx^{-1})}. \quad (2.44)$$

These representations are often referred as Rac and Di following [5].

$so(2, 4)$. Finally, in $d = 4$, the characters of long representations read

$$\chi_{\Delta,(\ell_1,\ell_2)}(q, x_1, x_2) = q^\Delta P_4(q, x_1, x_2) \chi_{(\ell_1,\ell_2)}^{so(4)}(x_1, x_2). \quad (2.45)$$

Since $so(4) \simeq so(3) \oplus so(3)$, the $so(4)$ character can be decomposed into the $so(3)$ ones as

$$\chi_{(\ell_1,\ell_2)}^{so(4)}(x_1, x_2) = \chi_{j_+}^{so(3)}(x_+) \chi_{j_-}^{so(3)}(x_-). \quad (2.46)$$

with

$$\ell_1 = j_+ + j_-, \quad \ell_2 = j_+ - j_-, \quad x_1 x_2 = x_+, \quad x_1 x_2^{-1} = x_-. \quad (2.47)$$

We get the explicit form for P_4 from (2.5) as

$$\begin{aligned} P_4(q, x_1, x_2) &= \frac{1}{(1 - q x_1) (1 - q x_1^{-1}) (1 - q x_2) (1 - q x_2^{-1})} \\ &= \frac{1}{\left(1 - q x_+^{\frac{1}{2}} x_-^{\frac{1}{2}}\right) \left(1 - q x_+^{\frac{1}{2}} x_-^{-\frac{1}{2}}\right) \left(1 - q x_+^{-\frac{1}{2}} x_-^{\frac{1}{2}}\right) \left(1 - q x_+^{-\frac{1}{2}} x_-^{-\frac{1}{2}}\right)}. \end{aligned} \quad (2.48)$$

With these, the character of semi-short representations are given by

$$\begin{aligned} \chi_{\mathcal{D}(\ell_1+2,(\ell_1,\ell_2))}(q, x_1, x_2) &= q^{\ell_1+2} P_4(q, x_1, x_2) \times \\ &\times \left(\chi_{j_+}^{so(3)}(x_+) \chi_{j_-}^{so(3)}(x_-) - q \chi_{j_+-\frac{1}{2}}^{so(3)}(x_+) \chi_{j_--\frac{1}{2}}^{so(3)}(x_-) \right). \end{aligned} \quad (2.49)$$

while that of the short representations by

$$\chi_{\mathcal{D}(1,(0,0))}(q, x_1, x_2) = q(1 - q^2) P_4(q, x_1, x_2), \quad (2.50)$$

$$\begin{aligned} \chi_{\mathcal{D}(s+1,(s,\pm s))}(q, x_1, x_2) &= q^{s+1} P_4(q, x_1, x_2) \times \\ &\times \left(\chi_s^{so(3)}(x_\pm) - q \chi_{s-\frac{1}{2}}^{so(3)}(x_\pm) \chi_{\frac{1}{2}}^{so(3)}(x_\mp) + q^2 \chi_{s-1}^{so(3)}(x_\pm) \right). \end{aligned} \quad (2.51)$$

We shall use this form of the character in computing corresponding one-loop diagrams.

2.2 Decomposition of singleton tensor product

Any physical Hilbert space \mathcal{H} of conformal field theory in d dimensions carries a unitary representation of $so(2, d)$, hence can be decomposed into the UIRs as

$$\mathcal{H} = \bigoplus_{\Delta, \ell} N_{\mathcal{D}(\Delta, \ell)}^{\mathcal{H}} \mathcal{D}(\Delta, \ell), \quad (2.52)$$

where $N_{\mathcal{D}(\Delta, \ell)}^{\mathcal{H}}$ are the multiplicities of the UIR, $\mathcal{D}(\Delta, \ell)$ in \mathcal{H} . Via the state-operator and AdS/CFT correspondences, each state carrying a UIR corresponds first to a CFT operator then to a bulk field. Since all UIRs of $so(2, d)$ are identified in the previous section, we can determine the bulk spectrum which corresponds to \mathcal{H} if the multiplicities $N_{\mathcal{D}(\Delta, \ell)}^{\mathcal{H}}$ are determined.

The typical way to extract the multiplicities from a given representation \mathcal{H} is the decomposition of its characters (or generalized partition function):

$$\chi_{\mathcal{H}}(q, \mathbf{x}) = \text{Tr}_{\mathcal{H}} \left[q^E x_1^{M_{12}} \dots x_h^{M_{h(h+1)}} \right] = \sum_{\Delta, \ell} N_{\mathcal{D}(\Delta, \ell)}^{\mathcal{H}} \chi_{\mathcal{D}(\Delta, \ell)}(q, \mathbf{x}), \quad (2.53)$$

where the character $\chi_{\mathcal{D}(\Delta, \ell)}$ is given in (2.9) with (2.4). The standard partition function $Z_{\mathcal{H}}(q)$ of the theory with Hilbert space \mathcal{H} is related to this character simply by

$$Z_{\mathcal{H}}(q) = \chi_{\mathcal{H}}(q, 1, \dots, 1), \quad (2.54)$$

but for the identification of the multiplicities, we need the full dependence in \mathbf{x} . In principle we can use an orthogonality relation between characters to extract the multiplicities [42], but in practice it is not simple to evaluate the necessary integrals. Instead, when the form of characters are simple enough, one can use more plain functional properties.

In the following, we shall derive various decomposition formulas by concentrating on the $so(2, 3)$ case. Note however that this will not limit our analysis of bulk quantum effect to the $d = 3$ case because we shall adopt a new method later on (hence, the rest of section 2.2 is not prerequisite for the following sections).

2.2.1 Laurent expansion of character

The key observation for the $so(2, d)$ character decomposition is that for a few lower dimensions the character take essentially a monomial form up to an overall function factor [43]. For $so(2, 3)$, the character of $\mathcal{V}(\Delta, \ell)$ satisfies

$$\frac{1 - x^{-1}}{P_3(q, x)} \chi_{\Delta, \ell}(q, x) = q^{\Delta} \left(x^{\ell} - x^{-\ell-1} \right), \quad (2.55)$$

the multiplicities of $\mathcal{V}(\Delta, \ell)$ can be obtained from Laurent expansion of the character as

$$N_{\mathcal{V}(\Delta, \ell)}^{\mathcal{H}} = \oint \frac{dx}{2\pi i x^{\ell+1}} \oint \frac{dq}{2\pi i q^{\Delta+1}} \frac{1 - x^{-1}}{P_3(q, x)} \chi_{\mathcal{H}}(q, x). \quad (2.56)$$

Hence, with this formula, for any reducible representation \mathcal{H} , once its character $Z_{\mathcal{H}}(q, x)$ is known, one can get the decomposition formula,

$$\mathcal{H} = \bigoplus_{\Delta, \ell} N_{\Delta, \ell}^{\mathcal{H}} \mathcal{V}(\Delta, \ell), \quad (2.57)$$

in terms of the LW representation $\mathcal{V}(\Delta, \ell)$. Note that in this case, the multiplicity $N_{\mathcal{V}(\Delta, \ell)}^{\mathcal{H}}$ might be negative integers. Afterwards, the decomposition (2.52) in terms of UIRs can be obtained by recollecting $\mathcal{V}(\Delta, \ell)$'s into $\mathcal{D}(\Delta, \ell)$'s, then the multiplicities $N_{\mathcal{D}(\Delta, \ell)}^{\mathcal{H}}$ are non-negative integers.

2.2.2 Oscillator representation

For $d = 3$ case, the oscillator representation of singletons [62, 63] has proven crucial in describing higher spin algebra as well as the Vasiliev's equations. It is also useful in

studying the Flato-Fronsdal theorem [5] and its extensions [64]. In this section, we show how the oscillator representation can be used to analyze the decomposition rule of the generic singleton tensor products,

$$\mathcal{H} = \text{Di}^{\otimes q} \otimes \text{Rac}^{\otimes p-q}, \quad (2.58)$$

where Di and Rac refers (following the standard terminology [5]) respectively the spinor and scalar singleton representation: $\text{Di} = \mathcal{D}(1, \frac{1}{2})$ and $\text{Rac} = \mathcal{D}(\frac{1}{2}, 0)$. This generalizes the Flato-Fronsdal theorem, to the cases of generic powers,

$$\text{Di}^{\otimes q} \otimes \text{Rac}^{\otimes p-q} = \bigoplus_{n=0}^{\infty} \bigoplus_{2s=0}^{\infty} N_{(m,n)}^{[q,p-q]} \mathcal{D}\left(s+n+\frac{p}{2}, s\right). \quad (2.59)$$

The explicit expression of $N_{(m,n)}^{[q,p-q]}$ can be identified using characters, whereas the oscillator representation provides a simple combinatoric account for the multiplicities:

The multiplicity $N_{(m,n)}^{[q,p-q]}$ of the tensor-product decomposition (2.59) is equal to the number of components in $\begin{array}{|c|} \hline m \\ \hline n \\ \hline \end{array}$ representation of $O(p)$, which are odd (and even) under the reflection of first q directions (last $p-q$ directions).

The singleton representations of $so(2, 3)$ algebra are realized by two sets of oscillators:

$$[a, a^\dagger] = 1 = [b, b^\dagger]. \quad (2.60)$$

In terms of these oscillators, the generators of $so(2) \oplus so(3)$ are given by

$$E = \frac{1}{2} (a^\dagger a + b^\dagger b + 1), \quad J_3 = \frac{1}{2} (a^\dagger a - b^\dagger b), \quad J_+ = a^\dagger b, \quad J_- = b^\dagger a. \quad (2.61)$$

The lowering operators are given by

$$M^-_- = a^2, \quad M^-_+ = b^2, \quad M^-_3 = ab, \quad (2.62)$$

and raising operators as the complex conjugate of the above. In this oscillator representation of $so(2, 3)$, two singletons, Di and Rac, are given as follows:

- Rac is the representation whose lowest weight state is the Fock vacuum $|0\rangle$ with $a|0\rangle = 0 = b|0\rangle$. Obviously the lowering operators (2.62) annihilate this state, and generic states of the Rac representation are constructed by acting raising operators on $|0\rangle$,

$$(M^+_+)^m (M^+_-)^n (M^+_3)^\ell |0\rangle = (a^\dagger)^{2m+\ell} (b^\dagger)^{2n+\ell} |0\rangle. \quad (2.63)$$

Hence, they have even number of oscillators. By acting E and J_3 on $|0\rangle$, we can immediately see that $|0\rangle$ defines $\mathcal{D}(\frac{1}{2}, 0)$.

- Di is the representation whose lowest weight state is the doublet $a^\dagger|0\rangle \oplus b^\dagger|0\rangle$. Since these states involve only one creation operator, they are still annihilated by the lowering operators (2.62). The generic states of Di have odd number of oscillators. By acting E and J_3 on $a^\dagger|0\rangle$ (the highest J_3 state), we find the doublet vacuum defines $\mathcal{D}(1, \frac{1}{2})$.

Higher spin algebra. This oscillator representation makes clear that the higher spin algebra is the maximal symmetry of singleton representations, namely the endomorphism of Rac (or Di) [63]. Since the singletons are constructed by acting the creation operators a^\dagger and b^\dagger on the Fock vacuum (or the doublet $a^\dagger|0\rangle \oplus b^\dagger|0\rangle$ for Di), any operators even orders in oscillators,

$$(a^\dagger)^m (b^\dagger)^n a^p b^q \quad [m+n+p+q \in 2\mathbb{N}], \quad (2.64)$$

belongs to the endomorphism, so the higher spin symmetry. The precise relation to the usual oscillators of high spin algebra reads

$$y_1 = a + b^\dagger, \quad y_2 = i(a^\dagger - b), \quad \bar{y}_{\dot{\alpha}} = (y_{\alpha})^\dagger. \quad (2.65)$$

Tensor product. We now consider tensor products of p singleton representations. They are realized by $p \times 2$ sets of oscillators:

$$[a_i, a_j^\dagger] = \delta_{ij} = [b_i, b_j^\dagger] \quad [i = 1, \dots, p]. \quad (2.66)$$

A generator T of $so(2, 3)$ is represented by

$$T = T_1 + \dots + T_p, \quad (2.67)$$

where T_i is the representation given only by the i -th oscillators. For instance, we have

$$E = \frac{1}{2}(N_a + N_b + p), \quad J_3 = \frac{1}{2}(N_a - N_b), \quad (2.68)$$

with

$$N_a = \sum_{i=1}^p a_i^\dagger a_i, \quad N_b = \sum_{i=1}^p b_i^\dagger b_i, \quad J_+ = \sum_{i=1}^p a_i^\dagger b_i, \quad M_-^- = \sum_{i=1}^p a_i^2. \quad (2.69)$$

We are looking for LW states in the k singleton tensor product space. Such LW states are not singlet under $so(3)$ so we focus only on the highest J_3 state among the $so(2, 3)$ LW states. Such a state with $(E, J_3) = (\Delta, s)$ is an eigenstate of the number operators (N_a, N_b) :

$$n_a = \Delta + s - \frac{p}{2}, \quad n_b = \Delta - s - \frac{p}{2}, \quad (2.70)$$

and can be expressed as

$$C_{n_a, n_b}(\mathbf{a}^\dagger, \mathbf{b}^\dagger) |0\rangle, \quad (2.71)$$

where $\mathbf{a}^\dagger = (a_1^\dagger, \dots, a_p^\dagger)$ and $\mathbf{b}^\dagger = (b_1^\dagger, \dots, b_p^\dagger)$ are p -dimensional vectors. The function $C_{n_a, n_b}(\mathbf{x}, \mathbf{y})$ satisfies

$$C_{n_a, n_b}(\lambda_a \mathbf{x}, \lambda_b \mathbf{y}) = \lambda_a^{n_a} \lambda_b^{n_b} C_{n_a, n_b}(\mathbf{x}, \mathbf{y}) \quad [\lambda_a, \lambda_b \in \mathbb{R}_{>0}]. \quad (2.72)$$

Then, in terms of this function, the vanishing M_-^- and J_+ conditions read

$$\partial_{\mathbf{x}}^2 C_{n_a, n_b}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \cdot \partial_{\mathbf{y}} C_{n_a, n_b}(\mathbf{x}, \mathbf{y}) = 0. \quad (2.73)$$

This defines actually two-row $O(k)$ Young diagram where the length of first and second rows are n_a and n_b , respectively. More precisely,

$$C_{n_a, n_b}(\mathbf{x}, \mathbf{y}) = C_{i_1 \dots i_{n_a}, j_1 \dots j_{n_b}} \frac{x_{i_1} \dots x_{i_{n_a}}}{n_a!} \frac{y_{j_1} \dots y_{j_{n_b}}}{n_b!}. \quad (2.74)$$

and

$$C_{i_1 \dots i_{n_a}, j_1 \dots j_{n_b}} \sim \begin{array}{|c|} \hline n_a \\ \hline n_b \\ \hline \end{array}. \quad (2.75)$$

Hence, the number of LW states with fixed n_a and n_b correspond to the dimensions of (n_a, n_b) Young diagrams, $\dim \left(\pi_{(n_a, n_b)}^{O(p)} \right)$:

$$\begin{aligned} \dim \left(\pi_{(n_a, n_b)}^{O(k)} \right) &= \frac{(n_a - n_b + 1)}{(n_a + 1)!} \frac{(k + n_a - 4)!}{n_b!} \frac{(k + 2n_a - 2)(k + n_a + n_b - 3)}{(k - 2)!} \\ &\quad \times \frac{(k + n_b - 5)!}{(k - 4)!} (k + 2n_b - 4). \end{aligned} \quad (2.76)$$

So far, we did not care where belong the LW states we found. Therefore, they correspond to tensor products of $\text{Di} \oplus \text{Rac}$,

$$(\text{Di} \oplus \text{Rac})^{\otimes p} = \bigoplus_{n=0}^{\infty} \bigoplus_{2s=0}^{\infty} \dim \left(\pi_{(n+2s, n)}^{O(p)} \right) \mathcal{D} \left(s + n + \frac{p}{2}, s \right). \quad (2.77)$$

Now, let us consider tensor products of Di 's and Rac 's. The Fock space of Di and Rac are constructed by odd and even numbers of oscillators, respectively. Hence, Di and Rac carry the alternating (-1) and the trivial $(+1)$ representations of $\mathbb{Z}_2 = \{1, \sigma\}$ generated by the oscillator sign flip operation $\sigma : (a^\dagger, b^\dagger) \mapsto (-a^\dagger, -b^\dagger)$. When considering the tensor product of k singletons, the group \mathbb{Z}_2 extends to $\mathbb{Z}_2^{\otimes p}$ which is the reflection subgroup of $O(p)$ and generated by

$$\sigma_i : (a_j^\dagger, b_j^\dagger) \mapsto (-1)^{\delta_{ij}} (a_j^\dagger, b_j^\dagger). \quad (2.78)$$

Renaming $\text{Rac} = S^{(+1)}$ and $\text{Di} = S^{(-1)}$, the tensor product $S^{(\epsilon_1)} \otimes S^{(\epsilon_2)} \otimes \dots \otimes S^{(\epsilon_p)}$ carries the $(\epsilon_1, \dots, \epsilon_p)$ representation of $\mathbb{Z}_2^{\otimes p}$. We consider the branching $O(p) \downarrow \mathbb{Z}_2^{\otimes p}$ of the (n_a, n_b) Young Diagram representation:

$$\pi_{(n_a, n_b)}^{O(p)} \Big|_{\mathbb{Z}_2^{\otimes p}} = \bigoplus_{\epsilon_1 = \pm 1} \dots \bigoplus_{\epsilon_p = \pm 1} N_{(n_a, n_b)}^{(\epsilon_1, \dots, \epsilon_p)} \pi_{(\epsilon_1, \dots, \epsilon_p)}^{\mathbb{Z}_2^{\otimes p}}, \quad (2.79)$$

where $N_{(n_a, n_b)}^{(\epsilon_1, \dots, \epsilon_p)} = N_{(n_a, n_b)}^{[q, p-q]}$ are the multiplicities (q is the number of Di 's) and it also gives the multiplicities of the tensor product decompositions. Therefore, the tensor product rule for $S^{(\pm 1)}$ can be written as (2.59). The explicit expression for $N_{(n+2s, n)}^{[q, p-q]}$ can be found using the combinatorics and its generating function turns out to coincide with the $so(2, 3)$ character. See appendix A for the details.

2.3 Single trace operators

So far, in considering tensor products of singletons, we have not taken any permutation symmetry of singletons into account. Let us denote such tensor-product spaces as

$$T^{(n)}(V) = V_1 \otimes \cdots \otimes V_n, \quad (2.80)$$

where V_i is the space of singleton representation and the subscript i is introduced to distinguish different copies of V . Notice however that the single trace operators (2.1) are invariant under cyclic permutations of $\partial^k \phi$'s due to the cyclicity of trace. This means that the space of the n -th order single trace operators corresponds not to $T^{(n)}(V)$ but to the subspace $T_{\text{cyc}}^{(n)}(V) \subset T^{(n)}(V)$,

$$T_{\text{cyc}}^{(n)}(V) = \bigoplus_{\pi \in C_n} V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)} = \bigoplus_{i=1}^n V_i \otimes V_{i+1} \otimes \cdots \otimes V_n \otimes V_1 \otimes \cdots \otimes V_{i-1}, \quad (2.81)$$

which is invariant under the actions of the cyclic group C_n . Therefore, when single trace operators admit symmetries such as cyclic permutations, we have to decompose the properly symmetrized tensor product of singletons into the UIRs of $so(2, d)$. Depending on the symmetry of the space, single trace operators have different symmetries, hence different tensor product should be used. In this paper, we focus on the CFT where the scalar field ϕ takes value in the adjoint representation of $SU(N)$: under the action of an $SU(N)$ element \mathbf{a} , the field transforms as

$$\phi \rightarrow \mathbf{a} \phi \mathbf{a}^{-1}. \quad (2.82)$$

In this case, the matrix ϕ does not admit any particular symmetry and its single trace operators, invariant under (2.82), only admit the cyclic symmetries. The character of the cyclic tensor-product space $T_{\text{cyc}}^{(n)}(V)$ is no more $[\chi_V(g)]^n$ but given by [36–43]

$$\chi_{\text{cyc}^n}(g) = \frac{1}{n} \sum_{k|n} \varphi(k) \left(\chi_V(g^k) \right)^{\frac{n}{k}} \quad [g^k = (q^k, x_1^k, \dots, x_h^k)], \quad (2.83)$$

where $k|n$ means the $k \in \{1, \dots, n\}$ which divides n and $\varphi(k)$ is the Euler totient function which counts the number of relative primes of k in $\{1, \dots, k\}$. The derivation of the above character is a result of the Polya's enumeration theorem. Hence, the character for the entire space of singlet trace operators is given by the sum of (2.83) over $n \geq 2$ (the $n = 1$ contribution drops out due to $\text{Tr}(\phi) = 0$). By changing the summation as $\sum_{n=1}^{\infty} \sum_{k|n} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty}$ with $n = mk$, we can perform the summation over m and get

$$\chi_{\text{adj}}(g) = \sum_{n=2}^{\infty} \chi_{\text{cyc}^n}(g) = -\chi_V(g) - \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log(1 - \chi_V(g^k)). \quad (2.84)$$

By decomposing this character into the UIRs, we can identify the spectrum of all single trace operators in the free scalar $SU(N)$ model. We remark here that the above formula relies on the infinite summation over m giving the log function. Hence, it turns out that the partition function develops singularities at finite non-zero values of β in contrast to the

vector models [65]. When the CFT_d is placed on $S^1 \times S^{d-1}$, β has the physical interpretation of inverse temperature and this divergence is related to the Hagedorn phase transition [20, 21]. In our case β is just a parameter which is useful for counting conformal dimensions of primaries. Nonetheless, we will see that these singularities play an important role in our vacuum energy computations and indeed introduce new ambiguities and subtleties not presented in the vector model CFT holography.

2.4 Explicit examples of decompositions

Let us conclude this section with a few examples of singleton tensor-product decompositions. We consider various bosonic tensor-products of Rac's and Di's up to order four. About the single trace operators, we confine ourselves to the scalar field cases, that involve only Rac's. In $O(p)$ Young diagrams, the number of boxes cannot exceed p for the first two columns. Therefore, the decomposition rules for two and three singleton tensor-products are particularly simple.

Two singletons. For the tensor-products of two singletons, we recover the well known result of Flato and Fronsdal:

$$\text{Rac}^{\otimes 2} = \bigoplus_{s=0}^{\infty} \mathcal{D}(s+1, s), \quad \text{Di}^{\otimes 2} = \mathcal{D}(2, 0) \oplus \bigoplus_{s=1}^{\infty} \mathcal{D}(s+1, s). \quad (2.85)$$

Let us now consider the adjoint model. Its operator spectrum makes use of the cyclic tensor-product,

$$T_{\text{cyc}}^{(2)}(\text{Rac}) = \bigoplus_{n=0}^{\infty} \mathcal{D}(2n+1, 2n), \quad (2.86)$$

which actually coincides with the result of the $O(N)$ -vector model.

Three singletons. For the tensor-products of three singletons, we obtain

$$\text{Rac}^{\otimes 3} = \bigoplus_{s=0}^{\infty} (s+1) \left[\mathcal{D}\left(s + \frac{3}{2}, s\right) \oplus \mathcal{D}\left(s + \frac{7}{2}, s+1\right) \right], \quad (2.87)$$

$$\text{Di}^{\otimes 2} \otimes \text{Rac} = \bigoplus_{s=0}^{\infty} (s+1) \left[\mathcal{D}\left(s + \frac{5}{2}, s\right) \oplus \mathcal{D}\left(s + \frac{5}{2}, s+1\right) \right]. \quad (2.88)$$

In the adjoint models, the spectrum of single trace operators is given by

$$T_{\text{cyc}}^{(3)}(\text{Rac}) = \bigoplus_{s=0}^{\infty} \left(s+1+2 \left[-\frac{s}{3} \right] \right) \left[\mathcal{D}\left(s + \frac{3}{2}, s\right) \oplus \mathcal{D}\left(s + \frac{7}{2}, s+1\right) \right], \quad (2.89)$$

where $[x]$ is the biggest integer not greater than x .

Four singletons. For the tensor-products of four singletons, we obtain

$$\begin{aligned} \text{Rac}^{\otimes 4} = & \bigoplus_{s=0}^{\infty} \frac{(1+s)(2+s)}{2} \mathcal{D}(s+2, s) \\ & \oplus \bigoplus_{s=0}^{\infty} \bigoplus_{n=1}^{\infty} \frac{(2n+2s+1)(2s+1) + 3(-1)^n}{4} \mathcal{D}(s+n+2, s), \end{aligned} \quad (2.90)$$

$$\begin{aligned} \text{Di}^{\otimes 2} \otimes \text{Rac}^{\otimes 2} = & \bigoplus_{s=0}^{\infty} \frac{(1+s)(2+s)}{2} \mathcal{D}(s+2, s) \\ & \oplus \bigoplus_{s=0}^{\infty} \bigoplus_{n=1}^{\infty} \frac{(2n+2s+1)(2s+1) - (-1)^n}{4} \mathcal{D}(s+n+2, s), \end{aligned} \quad (2.91)$$

$$\begin{aligned} \text{Di}^{\otimes 4} = & \bigoplus_{s=0}^{\infty} \frac{s(s-1)}{2} \mathcal{D}(s+2, s) \\ & \oplus \bigoplus_{s=0}^{\infty} \bigoplus_{n=1}^{\infty} \frac{(2n+2s+1)(2s+1) + 3(-1)^n}{4} \mathcal{D}(s+n+2, s). \end{aligned} \quad (2.92)$$

We can also obtain the spectrum of single trace operators for the $\text{SU}(N)$ adjoint model, but the formula become too lengthy and does not seem to be illuminating.

It is worth to note that the spectra with four fields contain the operators $\mathcal{D}(s+2, s-1)$ of twist $3 = (s+2) - (s-1)$ which can be interpreted as the one dual to the higher spin Goldstone modes [37–39, 41]. In fact, only the order four and six can give this contributions because the minimum twist $\tau = \Delta - s$ of the spectrum is larger than $n/2$ where n is the number of the conformal fields. The massless higher spin fields from the order two may acquire masses after combining with these modes from the order four or six. The fact that the Goldstone modes arises at these orders is the particularity of three dimensional scalar models where the conformal weight of scalar field is $1/2$ hence requires two or four more orders to give the Goldstone ones. In four dimensions where scalar field has weight one, only the order three operators can give the Goldstone modes. In particular, the operator dual to the scalar Goldstone can trigger a marginal deformation.

As one can see from the above results, the analytic formulas for the multiplicities of single trace operators become highly non-trivial as the number of Rac increases: the order four cyclic result would not fit in a single page. Consequently, the summation of physical quantities over such spectrum becomes practically intractable apart from a first few powers of Rac. In the following sections, we nevertheless makes use of this decomposition for some concrete calculations, but eventually proceed in a new approach.

3 AdS/CFT and bulk vacuum energy

In this paper, we aim to study the AdS theory which is dual to free scalar CFT in adjoint representation of $\text{SU}(N)$ by computing their one-loop vacuum energy. Before entering to the analysis of AdS side, let us remind the general picture behind this correspondence.

3.1 Holography for free matrix model CFTs

Let us first consider the CFT side which is described by a scalar action $S_{\text{CFT}}[\phi]$ where ϕ takes value in a matrix space of dimension N . For a $SU(N)$ adjoint model, $N = N^2 - 1$. Let $\{\mathcal{O}_{\Delta,\ell}^I\}$ denote the full set of single trace primary operators: Δ and ℓ are the labels for the conformal weight and spin,¹¹ and I is for the multiplicity. The connected correlators of single trace operators take the form,

$$\langle \mathcal{O}_{\Delta_1,\ell_1}^{I_1}(x_1) \cdots \mathcal{O}_{\Delta_n,\ell_n}^{I_n}(x_n) \rangle_{\text{con}} = \sum_k C_{\{\Delta_i,\ell_i\}k}^{\{I_i\}} G_{\{\Delta_i,\ell_i\}k}^k(\{x_i\}), \quad (3.1)$$

where the functions $G_{\{\Delta_i,\ell_i\}k}^k(\{x_i\})$ are the model independent tensor structures (labeled by k) allowed by the conformal symmetry, whereas the coefficients $C_{\{\Delta_i,\ell_i\}k}^{\{I_i\}}$ encode the particularity of the model.¹² In the convention that the operators are not normalized by N , the coefficients C (with the labels $\{\Delta_i,\ell_i\}k$ suppressed) admit a $1/N$ expansion:

$$C = N C^{(0)} + C^{(1)} + \frac{1}{N} C^{(2)} + \cdots. \quad (3.2)$$

A particular property of free CFTs is that the $1/N$ expansion becomes exact with the leading term alone: $C^{(n \geq 1)} = 0$. This property is a triviality from the CFT point of view, but it imposes a highly non-trivial requirement to the dual theory in AdS. Before moving to the AdS side, let us rephrase this property in terms of the generating function $F_{\text{CFT}}[h]$ of connected correlators. The latter admits the path-integral representation,

$$\exp(-F_{\text{CFT}}[h]) = \int D\phi \exp\left(-S_{\text{CFT}}[\phi] + \int d^d x \sum_{\Delta,\ell,I} \langle h_{\Delta,\ell}^I | \mathcal{O}_{\Delta,\ell}^I \rangle\right), \quad (3.3)$$

where $h_{\Delta,\ell}^I$ are the sources for single trace operators and $\langle \cdot | \cdot \rangle$ means the index contraction. Again, F_{CFT} of a generic matrix-model CFT admits a $1/N$ expansion:

$$F_{\text{CFT}} = N F_{\text{CFT}}^{(0)} + F_{\text{CFT}}^{(1)} + \frac{1}{N} F_{\text{CFT}}^{(2)} + \cdots, \quad (3.4)$$

but that of free CFTs have vanishing $F_{\text{CFT}}^{(n \geq 1)}$.

The AdS theory has the field contents $\{\varphi_{\Delta,\ell}^I\}$ (with the masses (2.19)) which are in one-to-one correspondence with the single trace operators. The conjecture states that the CFT physical quantity $F_{\text{CFT}}[h]$ coincides with the AdS one $\Gamma_{\text{AdS}}[h]$ given by

$$\exp(-\Gamma_{\text{AdS}}[h]) = \prod_{\Delta,\ell,I} \int_{\varphi|_{\partial\text{AdS}}=h} \mathcal{D}\varphi_{\Delta,\ell}^I \exp\left(-\frac{1}{G} S_{\text{AdS}}[\varphi]\right), \quad (3.5)$$

where the subscript $\varphi|_{\partial\text{AdS}}=h$ of the path-integral means that the fields are subject to the Dirichlet-like boundary condition,

$$\varphi_{\Delta,\ell}^I \sim z^\Delta h_{\Delta,\ell}^I \quad \text{as} \quad z \rightarrow 0, \quad (3.6)$$

¹¹In more than four spacetime dimensions, ℓ indicates the set of all quantum numbers required to specify the spin of the field.

¹²See [66–69] for explicit expressions for these tensor structures for three and four-point functions in CFT_3 and CFT_4 .

where z is the radial variable in the Poincaré coordinate of AdS whose boundary is located at $z = 0$. By denoting the unique classical solution smooth in the interior of (Euclidean) AdS and satisfying (3.6) as $\varphi_{\Delta,\ell}^I(h)$, we can split the fields into the classical background and the quantum fluctuation parts as $\varphi_{\Delta,\ell}^I = \varphi_{\Delta,\ell}^I(h) + \pi_{\Delta,\ell}^I$. Then, the G expansion of Γ_{AdS} ,

$$\Gamma_{\text{AdS}}[h] = \frac{1}{G} \Gamma_{\text{AdS}}^{(0)}[h] + \Gamma_{\text{AdS}}^{(1)}[h] + G \Gamma_{\text{AdS}}^{(2)}[h] + \cdots, \quad (3.7)$$

admits the diagrammatic interpretation that the 1PI scattering amplitudes for $\pi_{\Delta,\ell}^I$ are given by

$$\frac{\delta}{\delta h_1} \cdots \frac{\delta}{\delta h_n} \Gamma^{(\ell)}[h] \Big|_{h=0} = \text{Diagram} \quad (3.8)$$

The classical actions $S_{\text{AdS}}[\varphi]$ of the AdS theories dual to free scalar matrix models are not known, but the correspondence gives us various information about them:

- The massless sector of S_{AdS} coincides with the Vasiliev's theory, because free matrix models always have infinitely many conserved currents operators which are bilinear in ϕ and their correlators remains the same as the vector model case.
- The massive higher spin fields, which are the complement to the Vasiliev's spectrum in S_{AdS} , behave as matter sectors of the Vasiliev's higher spin gauge theory. Moreover, they consist of an infinite number of multiplets of Vasiliev's higher spin algebra: singleton can be regarded as the fundamental representation of Vasiliev algebra, and any its tensor products provide faithful representations of the algebra.
- Since free CFTs do not have any coupling constant, the only parameter that S_{AdS} involves is the cosmological constant or the radius R of AdS. This in turn combines with the gravitational constant G to form a dimensionless constant,

$$g := R^{1-d} G. \quad (3.9)$$

The latter should be related to the dimensionless parameter \mathbf{N} of the CFT.

- As we discussed above, the correlators of free CFTs have the same tensor structure for any value of \mathbf{N} . This implies that higher order loop corrections of Witten diagram at AdS dual theory should have the same tensor structure. This implies that the AdS dual theory cannot have different tensor structures for its Witten diagrams in different loop orders. Hence, they should be all proportional to the leading-order one,

$$G^\ell \Gamma_{\text{AdS}}^{(\ell)}[h] = n_\ell g^\ell \Gamma_{\text{AdS}}^{(0)}[h], \quad (3.10)$$

where n_ℓ are dimensionless constants. Furthermore, we have

$$\Gamma_{\text{AdS}}^{(0)}[h] = S_{\text{AdS}}[\varphi(h)]. \quad (3.11)$$

Hence, for a free CFT holography, we should have

$$\mathcal{N} F_{\text{CFT}}^{(0)}[h] = n(g) R^{1-d} S_{\text{AdS}}[\varphi(h)], \quad n(g) = \frac{1}{g} \left(1 + \sum_{\ell=1}^{\infty} n_{\ell} g^{\ell} \right). \quad (3.12)$$

By expanding the action S_{AdS} around AdS background in the power of fields as

$$S_{\text{AdS}}[\varphi] = S_0 + S_2[\varphi] + S_3[\varphi] + \cdots, \quad (3.13)$$

each interaction terms can be constructed in principle by comparing the corresponding Witten diagrams with the CFT correlators. For the quadratic and cubic terms S_2 and S_3 , it is sufficient to attach the boundary-to-bulk propagators to the vertices. In [70–73], all the cubic interactions for massive and massless symmetric higher spin fields have been constructed for their transverse and traceless pieces, which are enough for the on-shell calculations.¹³ The quartic term S_4 is more subtle as it requires to subtract infinitely many exchange diagrams, and a certain non-locality of interaction may start to appear from this order. In [75, 76], the scalar quartic interaction of Vasiliev’s theory has been identified in this way and it has been shown that its form is indeed a non-local one.¹⁴

3.2 Bulk vacuum energy and Zeta function

If the classical action S_{AdS} can be *constructed* from the holographic correspondence, then the conjecture becomes tautological in a sense at least at the classical level. A non-trivial test of the conjecture would be to calculate n -point functions starting from a given form of classical theory. The calculation of 3-pt correlators from Vasiliev’s theory corresponds to this case: the task has been carried out in [78] showing an agreement with the free Vector models. See [79–83] for further examinations based on a different technique.¹⁵ On the contrary to the Vasiliev’s theories, which are dual to the free vector models, the AdS theories dual to free matrix models are not known. The best guess is that it can be again described by a Vasiliev type equations extended by a certain matter sector. Anyway, as the classical theory is not known, the only available test of the correspondence would be the property (3.12): the quantum effects should be proportional to the semi-classical ones.

In the following, we consider the simplest case of the quantum effect: the one-loop ($\ell = 1$) diagram without any leg ($n = 0$) in (3.8), namely the vacuum energy. In principle, the vacuum energy cannot provide a rigorous test for (3.12) since it only returns a number rather than a tensor structure.¹⁶ However, since this number is typically a rather special

¹³Recently, the AdS cubic vertex which gives the 3-pt functions of the free scalar CFT has been determined in [74]. This vertex ought to be the metric-like form of the cubic interaction of Vasiliev’s theory.

¹⁴Once one accepts that a certain non-locality cannot be avoided, then one needs to distinguish a *good* non-locality from the *bad* ones whose introduction to classical actions spoils the predictability. See [77] for a recent discussion.

¹⁵For a certain type of 3pt correlators, a straightforward perturbative calculation involves some subtleties (noticed first in [78]) as reported in [84, 85] in relation with the allowed class of field redefinitions (see also [86, 87]). Hence, it would be fair to say that there still remains several issues to understand about the holography of Vasiliev’s theory.

¹⁶Strictly speaking, for the test, we need to consider the 3-pt correlators where different tensor structures start to enter. The 2-pt correlator is unique once the masses of fields are tuned by the conformal dimensions of the single trace operators.

rational or transcendental number, it may allow us to guess what the whole picture should look like. The vacuum energies of higher spin theories have been calculated for the vector model dualities [6–8] as well as some related extensions [9–17]. As we shall comment later, it has lead to an interesting guess on the relation between N and g .

The vacuum energy, $\Gamma_{\text{AdS}}^{(1)} := \Gamma_{\text{AdS}}^{(1)}[0]$ is given by

$$\exp(-\Gamma_{\text{AdS}}^{(1)}) = \prod_{\Delta, \ell, I} \int \mathcal{D}\pi_{\Delta, \ell}^I \exp\left(-\frac{1}{G} S_2[\pi]\right), \quad (3.14)$$

where the quadratic action S_2 (3.13) simply reduces to the sum of the quadratic actions for the fluctuation fields $\pi_{\Delta, \ell}^I$:

$$S_2[\pi] = \sum_{\Delta, \ell, I} S_{\Delta, \ell}[\pi^I]. \quad (3.15)$$

So far, we were using shorthand notations for $\pi_{\Delta, \ell}^I$ which are actually AdS tensor fields $\pi_{\mu(\ell_1), \dots, \mu(\ell_h)}^I$ carrying in general a mixed symmetry representation ℓ under the Lorentz symmetry. The precise form of the action $S_{\Delta, \ell}$ describing the $\mathcal{D}(\Delta, \ell)$ representation is non-trivial and generically requires to introduce traces and possibly other set of auxiliary fields. See [88–94] and references therein for the construction of such classical actions. For our purpose — which is to evaluate the vacuum energy diagrams — we need anyway to reduce the action to the traceless and transverse gauge where the form of the action is simplified to

$$S_{\Delta, \ell}[\pi_{\text{TT}}] = \frac{1}{2} \int d^{d+1}x \sqrt{g_{\text{AdS}}} \pi_{\text{TT}}^{\mu(\ell_1), \dots, \mu(\ell_h)} \mathcal{D}_{\Delta, \ell} \pi_{\text{TT} \mu(\ell_1), \dots, \mu(\ell_h)}, \quad (3.16)$$

where the differential operator $\mathcal{D}_{\Delta, \ell} = \square - M_{\Delta, \ell}^2$ is defined with the mass term (2.19). Here, we assumed that the corresponding representations are long ones. In case of short representations, the path-integral should be supplemented by the proper Jacobian (or ghost contribution) — which eventually amounts to extending the field content (labeled by I) to include the ghost fields with negative counting. The Laplacian operator \square depends on the tensor field that it acts on, specified by ℓ . Since the mass term will only shift the eigenvalues of $\mathcal{D}_{\Delta, \ell}$ from those of \square by an additive constant, we will mostly concern ourselves with the spectral problem for the Laplacian. The path-integral can be formally evaluated to give

$$\Gamma_{\text{AdS}}^{(1)} = \sum_{\Delta, \ell, I} -\frac{1}{2} \ln \det \mathcal{D}_{\Delta, \ell} = \sum_{\Delta, \ell} N_{\Delta, \ell}^{\mathcal{H}} \Gamma_{\Delta, \ell}^{(1)}, \quad (3.17)$$

where $N_{\Delta, \ell}^{\mathcal{H}} = \sum_I$ is the multiplicity of the fields $\mathcal{V}(\Delta, \ell)$ including the ghost contributions and $\Gamma_{\Delta, \ell}$ is the vacuum energy of the field corresponding to $\mathcal{V}(\Delta, \ell)$:

$$\Gamma_{\Delta, \ell}^{(1)} = -\frac{1}{2} \ln \det \mathcal{D}_{\Delta, \ell} = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr} \left[e^{-t \mathcal{D}_{\Delta, \ell}} \right]. \quad (3.18)$$

After the last equality, we have used the standard representation of the functional determinant assuming that $\mathcal{D}_{\Delta, \ell}$ has positive definite eigenvalues. The expression contains the

traced heat kernel,

$$K_{\Delta,\ell}(t) := \text{Tr} \left[e^{-t \mathcal{D}_{\Delta,\ell}} \right] = \int d^{d+1}x \sqrt{g_{\text{AdS}}} K_{\Delta,\ell}{}^{\mu}{}_{\mu}(x, x; t), \quad (3.19)$$

where μ is the shorthand notation for the indices $\mu(\ell_1), \dots, \mu(\ell_h)$ of $\chi^{\mu(\ell_1), \dots, \mu(\ell_h)}$ (which is again a shorthand notation (see footnote 9)). Given eigenfunctions $\psi_n^{\mu}(x)$ belonging to an eigenvalue E_n of the Laplacian \square , the heat kernel $K_{\Delta,\ell}{}^{\mu\nu}(x, y; t)$ is given by

$$K_{\Delta,\ell}{}^{\mu\nu}(x, y; t) = \sum_n \psi_n^{\mu}(x) \psi_n^{\nu}(y)^* e^{-t[E_n + M_{\Delta,\ell}^2]}. \quad (3.20)$$

For homogeneous spaces like spheres and hyperboloids (Euclidean AdS), the coincident heat kernel is independent of the position x , hence we find for AdS,

$$K_{\Delta,\ell}(t) = \text{Vol}_{\text{AdS}} k_{\Delta,\ell}(t), \quad k_{\Delta,\ell}(t) := K_{\Delta,\ell}{}^{\mu}{}_{\mu}(x, x; t), \quad (3.21)$$

where the volume of AdS, Vol_{AdS} , is a divergent quantity which requires a regularization. Besides this AdS IR divergence, we also have the usual UV divergences coming from the loop integrals. In the representation (3.18), it arises from the small t integral region, which corresponds to the short-distance heat propagation. This UV divergence can be regularized by a Mellin transform, namely the zeta function $\zeta_{\Delta,\ell}$, as

$$\Gamma_{\Delta,\ell}^{(1)}(\Lambda) = -\frac{1}{2} \Gamma(z) \zeta_{\Delta,\ell}(z), \quad \zeta_{\Delta,\ell}(z) = \int_0^\infty \frac{dt}{t} \frac{t^z}{\Gamma(z)} K_{\Delta,\ell}(t). \quad (3.22)$$

Collecting the contributions from the entire field content, we obtain

$$\Gamma_{\text{AdS}}^{(1)}(z) = -\frac{1}{2} \Gamma(z) \zeta_{\mathcal{H}}(z), \quad \zeta_{\mathcal{H}}(z) = \sum_{\Delta,\ell} N_{\Delta,\ell} \zeta_{\Delta,\ell}(z), \quad (3.23)$$

where \mathcal{H} is the physical Hilbert space of the theory (2.52). The dimensionless parameter z plays the role of UV cut-off:

$$\frac{1}{z} = \log(\Lambda_{\text{UV}} R), \quad (3.24)$$

with the UV cut-off scale Λ_{UV} and the AdS radius R , hence the renormalized vacuum energy of the theory is given by

$$\Gamma_{\text{AdS}}^{(1)\text{ren}} = -\frac{\zeta_{\mathcal{H}}(0)}{2} \log(\mu R) - \frac{\zeta'_{\mathcal{H}}(0)}{2}, \quad (3.25)$$

whereas the coefficient of the UV divergence is given by

$$\Gamma_{\text{AdS}}^{(1)\text{div}} = -\frac{\zeta_{\mathcal{H}}(0)}{2} \log \frac{\Lambda_{\text{UV}}}{\mu}. \quad (3.26)$$

Here, μ is the renormalization scale.

To summarize, the one-loop vacuum energy of the AdS theory is given by the function $\zeta_{\mathcal{H}}(z)$ (3.23), which is determined by the multiplicity of the spectrum $N_{\Delta,\ell}$ and the spectral zeta function $\zeta_{\Delta,\ell}$ (3.22). The calculation of the latter has been solved in great detail using group theoretic properties of AdS spaces [44–48], as we shall shortly review. About the

multiplicity $N_{\Delta,\ell}^{\mathcal{H}}$, we extract the spectrum $\{\varphi_{\Delta,\ell}^I\}$ of AdS fields by identifying it with the spectrum $\{\mathcal{O}_{\Delta,\ell}^I\}$ of conformal primaries in the dual CFT in the $N \rightarrow \infty$ limit. This is in any case a necessary condition for the validity of the given AdS/CFT duality.

Even then the problem of arriving at the total one-loop vacuum energy is still non-trivial since, as we saw, the general expression for the multiplicity of conformal primaries in a free CFT rapidly becomes very complicated as we start including contributions from higher and higher powers of singletons ϕ . We will shortly see this very explicitly for the case of AdS_4 .

4 AdS_4 with S^3 boundary

The first case we study is the computation of the one-loop vacuum energy in AdS_4 with S^3 boundary. This was also the first case that was considered in the higher spin/CFT duality computations for matching the one-loop free energy for Vasiliev higher spin theories with their CFT duals, the $U(N)$ and $O(N)$ vector models [6]. While the vector models have a relatively simple spectrum of conformal primaries, as the spectrum involves only the square of singletons, the $SU(N)$ adjoint model has a far richer spectrum as now arbitrarily high powers of singletons may be taken. In this respect, the case of AdS_4 is particularly simple because it cannot have any mixed symmetry tensor representations.

4.1 Zeta functions of AdS_4 fields

Fields in AdS_4 are labeled by their $so(2,3)$ quantum numbers (Δ, s) . The spectral zeta function for a field labeled by these quantum numbers is given by [46]

$$\zeta_{\Delta,s}(z) = \frac{1}{3} \int_0^\infty du u \tanh \pi u D_\Delta(z, u) S_s(u), \quad (4.1)$$

where $D_\Delta(z, u)$ and $S_s(u)$ are

$$D_\Delta(z, u) = \frac{1}{\left[u^2 + \left(\Delta - \frac{3}{2}\right)^2\right]^z}, \quad S_s(u) = \frac{2s+1}{2} \left[u + \left(\frac{2s+1}{2}\right)^2 \right]. \quad (4.2)$$

We shall provide two strategies for evaluating the zeta function of matrix model CFTs. First, we shall work explicitly with the spectra of conformal primaries obtained from considering powers of singletons and sum over the contributions of each such field. We shall find that this method quickly becomes prohibitive for two reasons. Firstly, as already encountered in the case of vector models [6, 7], these sums are naively divergent and need to be regularized. Secondly, the spectrum of conformal primaries becomes increasingly complicated as higher powers of singletons are taken. This leads us to consider an alternative method for computing the zeta function, using the character (or generalized partition function) of the dual CFT.

4.2 Vacuum energy from infinite series

We note that expressions for $\zeta_{\Delta,s}(0)$ and $\zeta'_{\Delta,s}(0)$ obtained from evaluating (4.1) are already known [45]. They are given by

$$\zeta_{\Delta,s}(0) = \frac{2s+1}{24} \left[\left(\Delta - \frac{3}{2}\right)^4 - \left(\frac{2s+1}{2}\right)^2 \left(2 \left(\Delta - \frac{3}{2}\right)^2 + \frac{1}{6} \right) - \frac{7}{240} \right], \quad (4.3)$$

and

$$\zeta'_{\Delta,s}(0) = \frac{2s+1}{3} \left[\frac{1}{8} \left(\Delta - \frac{3}{2} \right)^4 + \frac{1}{48} \left(\Delta - \frac{3}{2} \right)^2 + c_3 + \left(\frac{2s+1}{2} \right)^2 c_1 + \int_0^{\Delta-\frac{3}{2}} dx \left(\left(\frac{2s+1}{2} \right)^2 - x^2 \right) x \psi \left(x + \frac{1}{2} \right) \right]. \quad (4.4)$$

The constants c_n , where $n = 1, 3$, and the digamma function $\psi(x)$ are defined as

$$c_n = \int_0^\infty du \frac{2u^n \ln u}{e^{2\pi u} + 1}, \quad \psi(x) = \int_0^\infty dt \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right). \quad (4.5)$$

These expressions were used to evaluate $\zeta_{\Delta,s}(0)$ and $\zeta'_{\Delta,s}(0)$ for $\text{Rac}^{\otimes 2}/T_{\text{cyc}}^{(2)}(\text{Rac})$, the case relevant to the $U(N)/O(N)$ vector models in [6, 7].

For the vacuum-energy of the AdS theory under consideration, we need to use the tensor products $T_{\text{cyc}}^{(n)}$. However, as we have seen in section 2.4, the decomposition rule becomes quickly complicated as n increases. Hence, in the following, we will conduct the calculations first for the ‘toy models’ $\text{Rac}^{\otimes 3}$ and $\text{Rac}^{\otimes 4}$ in order to see how the methods adopted in [6, 7] can be extended to the $SU(N)$ adjoint model.

First, to compute the zeta function that includes the contribution of all the AdS fields corresponding to the conformal primaries contained in $\text{Rac}^{\otimes 3}$ we use the decomposition rule (2.88), and regulate the infinite sum over spins using zeta function regularisation. Firstly, it is easy to see that $\zeta_{\text{Rac}^{\otimes 3}}(0)$ vanishes:

$$\begin{aligned} \zeta_{\text{Rac}^{\otimes 3}}(0) &= \lim_{\alpha \rightarrow 0} \sum_{s=0}^{\infty} \left(\Delta - \frac{3}{2} \right)^{-\alpha} \zeta_{\Delta,s}(0) \\ &= \lim_{\alpha \rightarrow 0} \sum_{s=0}^{\infty} s^{-\alpha} \zeta_{s+\frac{3}{2},s}(0) + \lim_{\alpha \rightarrow 0} \sum_{s=0}^{\infty} (s+1)^{-\alpha} \zeta_{s+\frac{5}{2},s}(0) \\ &= -\frac{7}{13824} + \frac{7}{13824} = 0. \end{aligned} \quad (4.6)$$

Hence the logarithmically divergent part of the vacuum energy vanishes. We now consider the finite part of the one-loop free energy, contained in $\zeta'_{\text{Rac}^{\otimes 3}}(0)$. The nontrivial contribution to this comes solely from the digamma dependent part,

$$\begin{aligned} \zeta'_{\text{Rac}^{\otimes 3}}(0) &= \lim_{\alpha \rightarrow 0} \sum_{s=0}^{\infty} \int_0^\infty dt \left[s^{-\alpha} (s+1) \int_0^s dx + (s+1)^{-\alpha} s \int_0^{s+1} dx \right] \times \\ &\quad \times \frac{2}{3} \left(\frac{2s+1}{2} \right) \left[\left(\frac{2s+1}{2} \right)^2 x - x^3 \right] \left(\frac{e^{-t}}{t} - \frac{e^{-(x+\frac{1}{2})t}}{1 - e^{-t}} \right). \end{aligned} \quad (4.7)$$

After carrying out the integration over x and the summation over s , we obtain

$$\begin{aligned} \zeta'_{\text{Rac}^{\otimes 3}}(0) &= \int_0^\infty dt \frac{-2e^{-\frac{7t}{2}} - 6e^{-\frac{5t}{2}} - 6e^{-\frac{3t}{2}} - 2e^{-\frac{t}{2}}}{(1 - e^{-t})^4 t^4} \\ &\quad + \int_0^\infty dt \frac{-12e^{-\frac{7t}{2}} - 24e^{-\frac{5t}{2}} - 12e^{-\frac{3t}{2}}}{(1 - e^{-t})^5 t^3} \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\infty dt \frac{e^{-\frac{11t}{2}} - 23e^{-\frac{9t}{2}} - 362e^{-\frac{7t}{2}} - 362e^{-\frac{5t}{2}} - 23e^{-\frac{3t}{2}} + e^{-\frac{t}{2}}}{12(1-e^{-t})^6 t^2} \\
 & + \int_0^\infty dt \frac{5e^{-\frac{7t}{2}} + 18e^{-\frac{5t}{2}} + 5e^{-\frac{3t}{2}}}{2(1-e^{-t})^5 t}.
 \end{aligned} \tag{4.8}$$

By expressing these integrals in terms of the Lerch transcendent, we find that

$$\zeta'_{\text{Rac}^{\otimes 3}}(0) = -\frac{\ln 2}{128} - \frac{19\zeta(3)}{3840\pi^2} + \frac{25\zeta(5)}{256\pi^4} - \frac{63\zeta(7)}{512\pi^6}. \tag{4.9}$$

We next turn to the case of $\text{Rac}^{\otimes 4}$. In this case we will require the decomposition rule (2.90). Using this decomposition and (4.3), we see that

$$\begin{aligned}
 \zeta_{\text{Rac}^{\otimes 4}}(0) &= \lim_{\alpha \rightarrow 0} \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} \left(s+n+\frac{1}{2}\right)^{-\alpha} \frac{2n(2s+1) + 3(-1)^n + (2s+1)^2}{4} \zeta_{s+n+2,s}(0) \\
 &+ \lim_{\alpha \rightarrow 0} \sum_{s=0}^{\infty} \left(s+\frac{1}{2}\right)^{-\alpha} \frac{(s+1)(s+2)}{2} \zeta_{s+2,s}(0),
 \end{aligned} \tag{4.10}$$

where

$$\zeta_{s+n+2,s}(0) = \frac{2s+1}{24} \left[\left(s+n+\frac{1}{2}\right)^4 - \left(s+\frac{1}{2}\right)^2 \left(2\left(s+n+\frac{1}{2}\right)^2 + \frac{1}{6}\right) - \frac{7}{240} \right]. \tag{4.11}$$

To do the double summation we introduce a new parameter $m \equiv n+s$. After this replacement, we find that the first line of (4.10) gives

$$\lim_{\alpha \rightarrow 0} \sum_{m=1}^{\infty} \sum_{n=1}^m \left(m+\frac{1}{2}\right)^{-\alpha} \frac{2n[2(m-n)+1] + 3(-1)^n + [2(m-n)+1]^2}{4} \zeta_{m+2,m-n}(0), \tag{4.12}$$

which is equal to $-\frac{7963}{232243200}$. By a similar calculation, one can show that the second line gives $+\frac{7963}{232243200}$. Hence, the two contributions cancel each other and we again find that the UV divergent term $\zeta_{\text{Rac}^{\otimes 4}}(0)$ vanishes. The finite part of the vacuum energy is again given by the digamma piece of (4.4), and we get

$$\begin{aligned}
 & \zeta'_{\text{Rac}^{\otimes 4}}(0) \\
 &= \sum_{s=0}^{\infty} \int_0^\infty dt \int_0^{s+\frac{1}{2}} dx \left(s+\frac{1}{2}\right)^{1-\alpha} \frac{(s+1)(s+2)}{3} \left(\left(s+\frac{1}{2}\right)^2 x - x^3\right) \left(\frac{e^{-t}}{t} - \frac{e^{-(x+\frac{1}{2})t}}{1-e^{-t}}\right) \\
 &+ \sum_{m=1}^{\infty} \sum_{n=1}^m \int_0^\infty dt \int_0^{m+\frac{1}{2}} dx \left(m+\frac{1}{2}\right)^{-\alpha} \frac{2n[2(m-n)+1] + 3(-1)^n + [2(m-n)+1]^2}{12} \\
 &\times [2(m-n)+1] \left(\left(m-n+\frac{1}{2}\right)^2 x - x^3\right) \left(\frac{e^{-t}}{t} - \frac{e^{-(x+\frac{1}{2})t}}{1-e^{-t}}\right).
 \end{aligned} \tag{4.13}$$

We first carry out the integration over x and the summation over n . Next, the summation over m is carried out by Hurwitz zeta function regularization and we finally obtain

$$\begin{aligned}
 \zeta'_{\text{Rac}^{\otimes 4}}(0) = & \int_0^\infty dt \frac{2e^{-8t} + 2e^{-7t} - 6e^{-6t} - 6e^{-5t} + 6e^{-4t} + 6e^{-3t} - 2e^{-2t} - 2e^{-t}}{(1-e^{-t})^9 t^4} \\
 & + \int_0^\infty dt \frac{-e^{-8t} - 19e^{-7t} - 17e^{-6t} + 37e^{-5t} + 37e^{-4t} - 17e^{-3t} - 19e^{-2t} - e^{-t}}{(1-e^{-t})^9 t^3} \\
 & + \int_0^\infty dt \frac{e^{-8t} + 73e^{-7t} + 501e^{-6t} + 429e^{-5t} - 429e^{-4t} - 501e^{-3t} - 73e^{-2t} - e^{-t}}{6(1-e^{-t})^9 t^2} \\
 & + \int_0^\infty dt \frac{-40e^{-6t} - 184e^{-5t} - 184e^{-4t} - 40e^{-3t}}{(1-e^{-t})^9 t}. \tag{4.14}
 \end{aligned}$$

To obtain an analytic expression we again express the above in terms of the Lerch transcendent, and finally obtain

$$\zeta'_{\text{Rac}^{\otimes 4}}(0) = -\frac{29\zeta(3)}{2520\pi^2} - \frac{\zeta(5)}{60\pi^4} + \frac{\zeta(7)}{8\pi^6} - \frac{\zeta(9)}{8\pi^8}. \tag{4.15}$$

This completes the explicit evaluation of the zeta functions for the conformal primaries contained in $\text{Rac}^{\otimes 3}$ and $\text{Rac}^{\otimes 4}$. Even though we managed to get the results, it is hard to continue this method to higher order of singletons as the decomposition rules becomes quickly very complicated. Moreover if we consider the realistic matrix models instead of the toy models of $\text{Rac}^{\otimes n}$, even the order four level becomes highly non-trivial and it makes the computation more prohibitive. We now turn to an alternative approach to the same problem, which turns out to be of greater utility as it allows the powers of the singleton to increase.

4.3 Zeta function from character

We have seen above that while it is in principle possible to carry out a brute-force evaluation of the summation of the vacuum energies for higher powers of singletons, it is in practice quite prohibitive both due to increasing complexity of the spectrum and the careful regularization required at several steps in the calculation. We now introduce an alternative approach to the problem, based on the fact that the spectrum of conformal primaries is encoded in the corresponding character, namely the generalized partition function of the CFT. In particular, we would aim to write the zeta function corresponding to a given conformal primary in terms of the corresponding $so(2,3)$ character, and then formally carry out the sum over characters and multiplicities to write an expression for the zeta function for all the bulk fields in one go. As a starting point, we will rewrite the character (2.41) of $\mathcal{V}(\Delta, s)$ in the form

$$\chi_{\Delta,s}(\beta, \alpha) = \frac{e^{-(\Delta-\frac{3}{2})\beta} \sin\left(\frac{2s+1}{2}\alpha\right)}{\eta(\beta, \phi)}, \tag{4.16}$$

where $q = e^{-\beta}$ and $x = e^{i\alpha}$ are used in (2.41) and

$$\eta(\beta, \alpha) = 8 \sinh \frac{\beta}{2} \sin \frac{\alpha}{2} \left(\sinh^2 \frac{\beta}{2} + \sin^2 \frac{\alpha}{2} \right). \tag{4.17}$$

In order to relate $\zeta_{\Delta,s}$ to $\chi_{\Delta,s}$, we take the inverse Laplace transform of D_{Δ} as

$$D_{\Delta}(z, u) = \frac{\sqrt{\pi}}{\Gamma(z)} \int_0^{\infty} d\beta e^{-\beta(\Delta - \frac{3}{2})} \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(u\beta), \quad (4.18)$$

where we assumed $\Delta \geq \frac{3}{2}$. Next, we recast S_s as

$$S_s(u) = \frac{d}{d\alpha} \left(u^2 - \frac{d^2}{d\alpha^2} \right) \sin \left[\left(s + \frac{1}{2} \right) \alpha \right] \Big|_{\alpha=0}. \quad (4.19)$$

Using (4.18) and (4.19), the zeta function $\zeta_{\Delta,s}$ is related to the character $\chi_{\Delta,s}$ as

$$\zeta_{\Delta,s}(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} d\beta \left(\mu(z, \beta) + \nu(z, \beta) \frac{\partial^2}{\partial \alpha^2} \right) \chi_{\Delta,s}(\beta, \alpha) \Big|_{\alpha=0}, \quad (4.20)$$

where $\mu(z, \beta)$ and $\nu(z, \beta)$ are independent of Δ and s . Their expressions are given by

$$\begin{aligned} \mu(z, \beta) &= \frac{1}{3} \sinh \frac{\beta}{2} \left[\left(\sinh^2 \frac{\beta}{2} - 6 \right) f_1(z, \beta) + 4 \sinh^2 \frac{\beta}{2} f_3(z, \beta) \right] \\ \nu(z, \beta) &= -4 \sinh^3 \frac{\beta}{2} f_1(z, \beta), \end{aligned} \quad (4.21)$$

with $f_n(z, \beta)$ defined as

$$f_n(z, \beta) = \sqrt{\pi} \int_0^{\infty} du u^n \tanh \pi u \left(\frac{\beta}{2u} \right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta u). \quad (4.22)$$

Notice that $\mu(z, \beta)$ and $\nu(z, \beta)$ are simply fixed functions hence the relation (4.20) can be extended from $\mathcal{V}(\Delta, s)$ to any Hilbert space \mathcal{H} . In other words, it defines a universal relation between the spectral zeta function and the character. However, the identification of the functions $\mu(z, \beta)$ and $\nu(z, \beta)$ are non-trivial. When $z = 0$, the functions $\mu(z, \beta)$ and $\nu(z, \beta)$ have the simple form.

$$\mu(0, \beta) = \frac{2}{\beta} \frac{\cosh \frac{\beta}{2}}{\sinh \frac{\beta}{2}}, \quad \nu(0, \beta) = \frac{1}{\beta} \sinh \beta. \quad (4.23)$$

Let us notice that if we use $\mu(0, \beta)$ and $\nu(0, \beta)$ instead of $\mu(z, \beta)$ and $\nu(z, \beta)$, then the β integral will be divergent: the z -dependence was introduced precisely to regularize this UV divergence. Since we have freedom to choose convenient regularization scheme, we suggest a modification of zeta function (4.20) to

$$\tilde{\zeta}_{\mathcal{H}}(z) = \int_0^{\infty} d\beta \frac{\beta^{2z-1}}{\Gamma(2z)} \frac{\cosh \frac{\beta}{2}}{\sinh \frac{\beta}{2}} \left(1 + \sinh^2 \frac{\beta}{2} \partial_{\alpha}^2 \right) \chi_{\mathcal{H}}(\beta, \alpha) \Big|_{\alpha=0}, \quad (4.24)$$

which makes use of slightly different z -dependence which nevertheless does regularize the integral. Explicit comparison with the standard zeta function shows that their value at $z = 0$ coincide with each other implying that the log divergence of vacuum energy coincide. Moreover, they have the same first derivatives — proportional to the finite part of the vacuum energy — for the character which is even in β . This is the case for Rac itself as well as any tensor product of it. Therefore, for our purpose, the formula (4.24) is fully equivalent to the standard zeta function up to the z^2 order which is physically irrelevant. The details of the comparison can be found in appendix B.

4.4 A few tests

4.4.1 Vector models

Before analyzing the matrix models, we revisit the vector models as testing examples of the new method with $\tilde{\zeta}_{\mathcal{H}}$. The spectrum of non-minimal Vasiliev theory is given by the character

$$\chi_{\text{non-min}}(\beta, \alpha) = \chi_{\text{Rac}}^2(\beta, \alpha), \quad (4.25)$$

where the character of Rac is given in (2.44) in $(q = e^{-\beta}, x = e^{i\alpha})$, which can be re-expressed in (β, α) as

$$\chi_{\text{Rac}}(\beta, \alpha) = \frac{\cosh \frac{\beta}{2}}{\cosh \beta - \cos \alpha}. \quad (4.26)$$

By plugging (4.25) to (4.24), we find that the integrand itself vanishes, hence

$$\tilde{\zeta}_{\text{non-min}}(z) = 0. \quad (4.27)$$

As a result, both of the log divergent and finite parts are zero.

In the case of the minimal Vasiliev theory dual to free scalar $O(N)$ vector model, the character is given by

$$\chi_{\text{min}}(\beta, \alpha) = \frac{\chi_{\text{Rac}}^2(\beta, \alpha) + \chi_{\text{Rac}}(2\beta, 2\alpha)}{2}. \quad (4.28)$$

Since the $\chi_{\text{Rac}}^2 = \chi_{\text{non-min}}$ gives vanishing zeta function, the only contribution comes from the second term $\chi_{\text{Rac}}(2\beta, 2\alpha)/2$. After some manipulation, one can show that

$$\tilde{\zeta}_{\text{min}}(z) = 4^{-z} \tilde{\zeta}_{\text{Rac}}(z) = \tilde{\zeta}_{\text{Rac}}(z) + \mathcal{O}(z^2), \quad (4.29)$$

where $\tilde{\zeta}_{\text{Rac}}(z)$ reads

$$\tilde{\zeta}_{\text{Rac}}(z) = \zeta\left(2z - 2, \frac{1}{2}\right) + \frac{1}{4} \zeta\left(2z, \frac{1}{2}\right) = -\left[\frac{\ln 2}{4} - \frac{3\zeta(3)}{8\pi^2}\right]z + \mathcal{O}(z^2). \quad (4.30)$$

Interestingly, $\tilde{\zeta}_{\text{Rac}}$ reproduces the vacuum energy of the conformal scalar on the boundary.¹⁷ Therefore, the examples of vector models shows the agreement.

4.4.2 Toy models: $\text{Rac}^{\otimes 3}$ and $\text{Rac}^{\otimes 4}$

Let us move to the toy model examples of $\text{Rac}^{\otimes n}$. The corresponding vacuum energies have been calculated in section 4.2, and in below, we shall check whether the new method can reproduce the result correctly. The character for $\text{Rac}^{\otimes n}$ is simply

$$\chi_{\text{Rac}^{\otimes n}}(\beta, \alpha) = \chi_{\text{Rac}}^n(\beta, \alpha), \quad (4.31)$$

and the corresponding spectral zeta function leads to the integral,

$$\tilde{\zeta}_{\text{Rac}^{\otimes n}}(z) = -\frac{n-2}{2} \int_0^\infty d\beta \frac{\beta^{2z-1}}{\Gamma(2z)} e^{-\frac{n}{2}\beta} \frac{(1+e^{-\beta})^{n+1}}{(1-e^{-\beta})^{2n+1}}. \quad (4.32)$$

¹⁷See [13] for more discussion on this occurrence between the partition functions in different dimensions.

For specific values $n = 3$ and $n = 4$, the integrals can be evaluated by Taylor expanding the integrand in $e^{-\beta}$ and eventually resumming. In the end, we obtain

$$\begin{aligned}\tilde{\zeta}_{\text{Rac}^{\otimes 3}}(z) &= \frac{\zeta(2z, \frac{3}{2})}{128} - \frac{19\zeta(2z-2, \frac{3}{2})}{1440} - \frac{5\zeta(2z-4, \frac{3}{2})}{72} - \frac{\zeta(2z-6, \frac{3}{2})}{90} \\ &= z \left(-\frac{\ln 2}{128} - \frac{19\zeta(3)}{3840\pi^2} + \frac{25\zeta(5)}{256\pi^4} - \frac{63\zeta(7)}{512\pi^6} \right) + \mathcal{O}(z^2),\end{aligned}\quad (4.33)$$

$$\begin{aligned}\tilde{\zeta}_{\text{Rac}^{\otimes 4}}(z) &= \frac{29\zeta(2z-2)}{1260} - \frac{\zeta(2z-4)}{90} - \frac{\zeta(2z-6)}{90} - \frac{\zeta(2z-8)}{1260} \\ &= z \left(-\frac{29\zeta(3)}{2520\pi^2} - \frac{\zeta(5)}{60\pi^4} + \frac{\zeta(7)}{8\pi^6} - \frac{\zeta(9)}{8\pi^8} \right) + \mathcal{O}(z^2).\end{aligned}\quad (4.34)$$

By comparing these with the results (4.9) and (4.15) obtained in section 4.2 by summation over the field content, we find these results exactly matches to those. One can see that compared to the calculations in section 4.2, the new integral representation $\tilde{\zeta}_{\mathcal{H}}(z)$ requires a considerably shorter calculation.

4.5 Vacuum energy for the AdS dual of $\text{SU}(N)$ adjoint model

Now, let us turn to the vacuum energy of the AdS theory dual to the $\text{SU}(N)$ adjoint matrix model.

4.5.1 Vacuum energies for the first few Regge trajectories

To begin with, we calculate the spectral zeta function for the first few Regge trajectories, that is, for the first few orders in the power of singleton ϕ . In the end, the total vacuum energy is the sum of the ones of the order from two to infinity.

Order two. At the order two, the field content coincides with that of the minimal Vasiliev theory hence their vacuum energies coincides with that of Rac:

$$\tilde{\zeta}_{\text{cyc}^2}'(0) = -\frac{\ln 2}{4} + \frac{3\zeta(3)}{8\pi^2} \simeq -0.127614. \quad (4.35)$$

Order three. At the order three, the $\text{SU}(N)$ adjoint model corresponds to the cyclic character,

$$\chi_{\text{cyc}^3}(\beta, \alpha) = \frac{\chi_{\text{Rac}}^3(\beta, \alpha) + 2\chi_{\text{Rac}}(3\beta, 3\alpha)}{3}, \quad (4.36)$$

and gives the zeta function,

$$\begin{aligned}\tilde{\zeta}_{\text{cyc}^3}(z) &= \frac{\zeta(2z, \frac{3}{2})}{1152} - \frac{19\zeta(2z-2, \frac{3}{2})}{17280} - \frac{5\zeta(2z-4, \frac{3}{2})}{216} - \frac{\zeta(2z-6, \frac{3}{2})}{270} \\ &\quad + 2^{2z} 3^{-2z-1} [\zeta(2z) + \zeta(2z-2)] - 3^{-2z-1} [\zeta(2z) + 4\zeta(2z-2)] \\ &\quad + 3^{-2z-3} \frac{57}{32} \left[\zeta\left(2z, \frac{1}{6}\right) + \zeta\left(2z, \frac{5}{6}\right) - 8\zeta\left(2z-1, \frac{1}{6}\right) + 8\zeta\left(2z-1, \frac{5}{6}\right) \right] \\ &\quad + 3^{-2z-1} \left[\zeta\left(2z-2, \frac{1}{6}\right) + \zeta\left(2z-2, \frac{5}{6}\right) - 3\zeta\left(2z-3, \frac{1}{6}\right) + 3\zeta\left(2z-3, \frac{5}{6}\right) \right].\end{aligned}\quad (4.37)$$

It is free from the UV divergence because $\tilde{\zeta}_{\text{cyc}^3}(0) = 0$. However, the finite part does not vanish but gives

$$\begin{aligned}\tilde{\zeta}_{\text{cyc}^3}'(0) &= -\frac{43 \ln 2}{128} + \frac{1487 \zeta(3)}{3840 \pi^2} + \frac{25 \zeta(5)}{768 \pi^4} - \frac{21 \zeta(7)}{512 \pi^6} + \frac{4 \pi}{27 \sqrt{3}} - \frac{19 \psi^{(1)}(\frac{1}{3})}{72 \sqrt{3} \pi} + \frac{\psi^{(3)}(\frac{1}{3})}{96 \sqrt{3} \pi^3} \\ &\simeq -0.311588,\end{aligned}\tag{4.38}$$

where $\psi^{(n)}$ is the polygamma function of order n . The order three trajectory has more than two times the vacuum energy than the order two. Let us also note that the particular property of the order two does not seem to continue to the order three.

Order four. At the level order, the $\text{SU}(N)$ adjoint model character is given by

$$\chi_{\text{cyc}^4}(\beta, \alpha) = \frac{\chi_{\text{Rac}}^4(\beta, \alpha) + \chi_{\text{Rac}}^2(2\beta, 2\alpha) + 2\chi_{\text{Rac}}(4\beta, 4\alpha)}{4},\tag{4.39}$$

and the corresponding zeta function reads

$$\begin{aligned}\tilde{\zeta}_{\text{cyc}^4}(z) &= \frac{5 \zeta(2z)}{16} - \frac{47 \zeta(2z-2)}{2520} - \frac{23 \zeta(2z-4)}{720} - \frac{\zeta(2z-6)}{90} - \frac{\zeta(2z-8)}{1260} \\ &\quad + 2^{-2z} \left[\frac{3 \zeta(2z)}{2} + \frac{4 \zeta(2z-2)}{3} + \frac{2 \zeta(2z-4)}{3} \right] \\ &\quad + 2^{-4z} \left[\frac{5 \zeta(2z-1, \frac{3}{4})}{6} - \frac{5 \zeta(2z-1, \frac{1}{4})}{6} + \frac{8 \zeta(2z-3, \frac{3}{4})}{3} - \frac{8 \zeta(2z-3, \frac{1}{4})}{3} - \frac{\zeta(2z)}{2} \right].\end{aligned}\tag{4.40}$$

Again, one can show that the vacuum energy is finite, $\tilde{\zeta}_{\text{cyc}^4}(0) = 0$, and the finite part is given by

$$\begin{aligned}\tilde{\zeta}_{\text{cyc}^4}'(0) &= -\frac{25 \ln 2}{64} - \frac{4573 \zeta(3)}{20160 \pi^2} + \frac{457 \zeta(5)}{1920 \pi^4} + \frac{\zeta(7)}{32 \pi^6} - \frac{\zeta(9)}{32 \pi^8} + \frac{\pi}{32} - \frac{5 \psi^{(1)}(\frac{1}{4})}{96 \pi} + \frac{\psi^{(3)}(\frac{1}{4})}{384 \pi^3} \\ &\simeq -0.353518.\end{aligned}\tag{4.41}$$

The vacuum energy of the fourth order trajectory is increased again compared to the order three, but only by a small amount.

Higher orders. In order to see a pattern, we can proceed with a few more orders: the vacuum energies of the first 8 order Regge trajectories are plotted in figure 1. One can see that the energies exhibit a rough linear growth. Assuming that the pattern continues, we can conclude that the total vacuum energy — the sum of all vacuum energies in different trajectories — will be divergent. Therefore, we need to perform a regularization yet another time. If we had analytic expressions for the vacuum energies for different orders, we may consider various possible regularizations, but unfortunately we could not find the analytic formula.

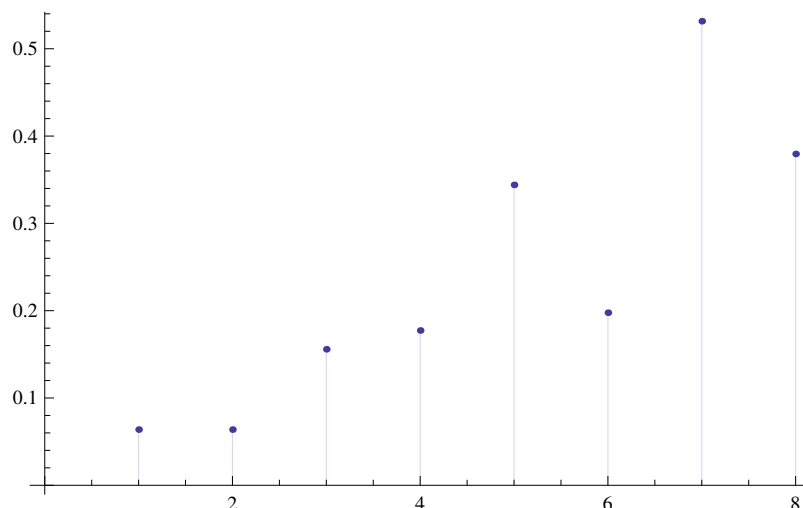


Figure 1. Vacuum energies of AdS_4 fields in different orders.

4.5.2 Vacuum energies in different slices

In the previous section, we have calculated the vacuum energy of the AdS theory for the first few Regge trajectories, which show a growing pattern. In this section, we will consider the vacuum energy as a difference series. As we have explained in section 2.3, in the limit $N \rightarrow \infty$, the characters for the matrix models can be partly resummed ending up with a logarithmic function with Euler totient function as (2.84). By expressing it as

$$\chi_{\text{adj}} = -\chi_{\text{Rac}} + \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \chi_{\log,k}, \quad (4.42)$$

with

$$\chi_{\log,k}(\beta, \alpha) = -\log[1 - \chi_{\text{Rac}}(k\beta, k\alpha)], \quad (4.43)$$

we can focus on the spectral zeta function corresponding to $\chi_{\log,k}$,

$$\begin{aligned} \tilde{\zeta}_{\log,k}(z) = & - \int_0^\infty d\beta \frac{\beta^{2z-1}}{\Gamma(2z)} \left[\frac{\cosh \frac{\beta}{2}}{\sinh \frac{\beta}{2}} \log \left(1 - \frac{\cosh \frac{k\beta}{2}}{2 \sinh^2 \frac{k\beta}{2}} \right) \right. \\ & \left. - \frac{k^2}{2} \sinh \beta \left(\frac{1}{1 + \cosh \frac{k\beta}{2} - \cosh k\beta} + \frac{1}{\cosh k\beta - 1} \right) \right]. \end{aligned} \quad (4.44)$$

Working with a fixed value of k corresponds to considering the full vacuum energy as a series in k . To distinguish with the previous organization of the spectrum of primaries into Regge trajectories, we refer to all the primaries appearing with a fixed k as belonging to the same ‘slice’. The evaluation of the above integral is technically prohibitive, but one can easily check that the above two functions vanish when $z = 0$, implying that the vacuum energy is free from the logarithmic UV divergence. The renormalized vacuum energy corresponding to the first derivative of $\tilde{\zeta}_{\log,k}$ is not easily accessible by analytic method. We can nevertheless proceed numerically (see appendix B), but the validity of such result is not fully clear.

In the above, we were attacking the computation of the vacuum energy of the AdS theory dual to $SU(N)$ adjoint model as two different series: one as a series of the order of fields (considered in section 4.5.1), and the other as a series with a fixed Euler totient function (considered in section 4.5.2). In both case, we could obtain the contributions for a few low orders, but an analytic expression for whole sequence was not available. At this point, one may wonder whether we can consider the full character where different k contributions are summed up. In a sense, this idea seems to be in the continuation of our reasoning: whenever we face a series which is divergent, we consider directly the character which originates the series. However at this time, the situation is different. Clearly, the full character for the adjoint model has an infinite number of singularities at $\beta = \frac{\beta_c}{k}$, where β_c is the singular point for the $k = 1$ part. Moreover, around $\beta = 0$, the singularities corresponding higher k values are accumulated making the character highly non-analytic around $\beta = 0$ point. Hence, if we consider the full character function, it seems to be impossible to control the divergence arising from the small β region because its non-analyticity is severe. Therefore, the only well-defined ways to address the adjoint models would be what we have been considered here: as a series in the order of trajectories or as a series in the order of ‘slices’.

5 AdS₅ with S^4 boundary

We now extend the approach of writing the bulk zeta function in terms of the character — that is, the generalized partition function — of the boundary CFT to the case of AdS₅. This is the first instance where fields of mixed symmetry make an appearance. Since the case of completely symmetric fields is well understood [46] and has already been applied to this context in [7], we omit that discussion and consider directly the most general case of mixed symmetry fields. The expressions obtained in the next section for the zeta function have previously been obtained in appendix C of [9]. Nonetheless, it is useful to review this expression as it will prove important for the subsequent analysis.

5.1 Zeta functions of AdS₅ fields

We start from an AdS₅ field that labelled by the $so(2,4)$ quantum numbers $(\Delta, (\ell_1, \ell_2))$. Furthermore, we assume this to be a long representation of the conformal algebra, as characters of (semi-)short representations can be written as sums and differences of characters of long representations. It was observed in [44–48] that for many classes of fields — including symmetric transverse traceless tensors, spinors, and p -forms — the coincident heat kernel for the Laplacian on AdS_{2n+1} is simply obtained by an analytic continuation¹⁸ of the corresponding quantity on S^{2n+1} . Inspired by these lessons, we will compute the heat kernel over generic mixed symmetry fields on AdS₅ by relating it to a heat kernel over S^5 via analytic continuation. We now describe how this is done, using the following results from harmonic analysis on spheres. Let \mathcal{S} be the space of fields on a five-sphere $S^5 = SO(6)/SO(5)$, then the space \mathcal{S} carries a UIR of $SO(5)$. The eigenvalues of the Laplacian acting on \mathcal{S} are determined by the quadratic Casimirs $Q_{\mathcal{R}}$ of the UIRs \mathcal{R} of $SO(6)$

¹⁸Subtleties related to even dimensional spheres and hyperboloids are discussed at length in [44, 46–48].

whose restriction to $SO(5)$ include the representation \mathcal{S} . By imposing irreducibility conditions (such as transversality or tracelessness) on the fields, one can further constrain the permitted set of \mathcal{R} s. Then by definition, the traced and integrated heat kernel is given by

$$K_{\mathcal{S}}(t) = \sum_{\mathcal{R}|\mathcal{S}} d_{\mathcal{R}} e^{-t Q_{\mathcal{R}}}, \quad (5.1)$$

where $\mathcal{R}|\mathcal{S}$ means that all the UIR \mathcal{R} of $SO(6)$ whose restriction to $SO(5)$ include \mathcal{S} . The degeneracy of the eigenvalue, $d_{\mathcal{R}}$, is given by the dimension of the representation \mathcal{R} . Using the homogeneity of the sphere, we conclude that the corresponding coincident heat kernel is given by

$$k_{\mathcal{S}}(t) = \sum_{\mathcal{R}|\mathcal{S}} \frac{d_{\mathcal{R}}}{\text{Vol}_{S^5}} e^{-t Q_{\mathcal{R}}}. \quad (5.2)$$

More concretely, we consider the space of the fields on S^5 which transforms in

$$\mathcal{S} = (\ell_1, \ell_2) \quad [\ell_1 \geq \ell_2 \geq 0], \quad (5.3)$$

representation of $SO(5)$ and satisfy irreducibility conditions. Then, the $SO(6)$ UIRs which can branch into \mathcal{S} are

$$\mathcal{R} = (\ell_0, \ell_1, \pm \ell_2) \quad [\ell_0 \geq \ell_1]. \quad (5.4)$$

Hence, for a given \mathcal{S} , namely (ℓ_1, ℓ_2) , we have one free integer parameter ℓ_0 for \mathcal{R} and the choice of the sign in $\pm \ell_2$. The dimension $d_{\mathcal{R}}$ of the above representation is

$$d_{\mathcal{R}} = d_{(\ell_0, \ell_1, \pm \ell_2)} = \frac{(\ell_0 + 2)^2 - (\ell_1 + 1)^2}{2^2 - 1^2} \frac{(\ell_0 + 2)^2 - \ell_2^2}{2^2 - 0^2} \frac{(\ell_1 + 1)^2 - \ell_2^2}{1^2 - 0^2}, \quad (5.5)$$

and it does not depend on the sign in $\pm \ell_2$. Moreover the quadratic Casimirs also do not depend on this sign: $Q_{(\ell_0, \ell_1, +\ell_2)} = Q_{(\ell_0, \ell_1, -\ell_2)}$, hence the coincident heat kernel $k_{(\ell_1, \ell_2)}(t)$ receives the same contribution twice, once from $(\ell_0, \ell_1, +\ell_2)$ and the other time from $(\ell_0, \ell_1, -\ell_2)$. By imposing a duality condition on \mathcal{S} , one can also restrict \mathcal{R} to one of these two.

The corresponding computation for AdS_5 is more subtle because $AdS_5 = SO(1, 5)/SO(5)$, and $SO(1, 5)$ being non-compact admits infinite dimensional unitary irreducible representations. It has been explicitly demonstrated for many classes of fields in [44–48] that the coincident heat kernel (5.2) on S^5 may be analytically continued to that on AdS_5 via

$$\ell_0 \mapsto i u - 2 \quad [0 < u < \infty]. \quad (5.6)$$

Moreover the sum over ℓ_0 gets mapped to an integral over u and t becomes $-t$, and we obtain

$$k_{(\ell_1, \ell_2)}(t) = \int_0^\infty du \mu(u; \ell_1, \ell_2) e^{t Q_{(i u - 2, \ell_1, \ell_2)}}, \quad (5.7)$$

where the measure,

$$\mu(u; \ell_1, \ell_2) = \frac{d_{(i u - 2, \ell_1, \ell_2)}}{\text{Vol}_{S^5}} = \frac{(u^2 + (\ell_1 + 1)^2) (u^2 + \ell_2^2) (\ell_1 + \ell_2 + 1) (\ell_1 - \ell_2 + 1)}{12 \text{Vol}_{S^5}}, \quad (5.8)$$

is known as the Plancherel measure and intuitively corresponds to the degeneracy of the eigenvalue $Q_{(i u - 2, \ell_1, \ell_2)}$. It was observed in [95] that to reproduce the thermal partition function of a conformal primary $\mathcal{D}(\Delta, (\ell_1, \ell_2))$, one has to consider quadratic fluctuations of fields carrying the representation $(i u - 2, \ell_1, \ell_2)$ of $\text{SO}(6)$, but the eigenvalues of the kinetic operator should be shifted such that the coincident heat kernel of the kinetic operator is replaced by

$$k_{\Delta, (\ell_1, \ell_2)}(t) = \int_0^\infty du \mu(u; \ell_1, \ell_2) e^{-t[u^2 + (\Delta - 2)^2]}. \quad (5.9)$$

On going through these replacements and taking the Mellin transform, we arrive at the following expression for the zeta function corresponding to the coincident heat kernel.

$$\zeta_{\Delta, (\ell_1, \ell_2)}(z) = \int_0^\infty \frac{dt}{t} \frac{t^z}{\Gamma(z)} K_{\Delta, (\ell_1, \ell_2)}(t) = \frac{\text{Vol}_{\text{AdS}_5}}{3\pi^3} \int_0^\infty du D_\Delta(z, u) S_{(\ell_1, \ell_2)}(u), \quad (5.10)$$

where D_Δ and $S_{(\ell_1, \ell_2)}$ are given by

$$D_\Delta(z, u) = \frac{1}{[u^2 + (\Delta - 2)^2]^z}, \quad (5.11)$$

$$S_{(\ell_1, \ell_2)}(u) = \frac{\ell_1 + \ell_2 + 1}{2} \frac{\ell_1 - \ell_2 + 1}{2} (u^2 + (\ell_1 + 1)^2) (u^2 + \ell_2^2). \quad (5.12)$$

For the case of symmetric tensors, i.e. $\ell_2 = 0$, this expression agrees with [7]. For the later analysis, it is convenient to work also with the $su(2) \oplus su(2)$ label $[j_+, j_-]$ together with the $so(4)$ one (ℓ_1, ℓ_2) . The relation between $[j_+, j_-]$ and (ℓ_1, ℓ_2) is given by (2.47), and the function $S_{[j_+, j_-]} = S_{(\ell_1, \ell_2)}$ reads

$$S_{[j_+, j_-]}(u) = \frac{2j_+ + 1}{2} \frac{2j_- + 1}{2} (u^2 + (j_+ + j_- + 1)^2) (u^2 + (j_+ - j_-)^2). \quad (5.13)$$

Finally, the volume of AdS_5 can be regularized to

$$\text{Vol}_{\text{AdS}_5} = \pi^2 \log(\mu R), \quad (5.14)$$

as in [96]. Hence, the IR regularized vacuum energy in AdS_5 is proportional to the logarithm of the AdS radius R and the holographic renormalization scale μ . Hereafter, we suppress the dependence of μ as it always appears with R .

5.2 Zeta function from character

Next we will show how this zeta function may be written in terms of the character (2.45) of the representation $\mathcal{V}(\Delta, [j_+, j_-])$. The latter is given in the variables $(\beta, \alpha_+, \alpha_-)$, with $q = e^{-\beta}$ and $x_\pm = e^{i\alpha_\pm}$, by

$$\chi_{\Delta, [j_+, j_-]}(\beta, \alpha_+, \alpha_-) = \frac{e^{-(\Delta-2)\beta} \sin\left(\frac{2j_++1}{2}\alpha_+\right) \sin\left(\frac{2j_-+1}{2}\alpha_-\right)}{\eta(\beta, \alpha_+, \alpha_-)}, \quad (5.15)$$

where

$$\eta(\beta, \alpha_+, \alpha_-) = 4 \sin \frac{\alpha_+}{2} \sin \frac{\alpha_-}{2} \left(\cosh \beta - \cos \left(\frac{\alpha_+ + \alpha_-}{2} \right) \right) \left(\cosh \beta - \cos \left(\frac{\alpha_+ - \alpha_-}{2} \right) \right). \quad (5.16)$$

In the formula (5.10) for the zeta function, we replace the factor D_Δ (5.11) by

$$D_\Delta(z, u) = \frac{\sqrt{\pi}}{\Gamma(z)} \int_0^\infty d\beta e^{-(\Delta-2)\beta} \left(\frac{\beta}{2u} \right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(u\beta), \quad (5.17)$$

and the factor $S_{[j_+, j_-]}$ (5.12) by

$$S_{[j_+, j_-]}(u) = \partial_{\alpha_+} \partial_{\alpha_-} (u^2 - (\partial_{\alpha_+} + \partial_{\alpha_-})^2) (u^2 - (\partial_{\alpha_+} - \partial_{\alpha_-})^2) \times \\ \times \sin\left(\frac{2j_+ + 1}{2} \alpha_+\right) \sin\left(\frac{2j_- + 1}{2} \alpha_-\right) \Big|_{\alpha_\pm=0}. \quad (5.18)$$

The left hand sides of (5.17) and (5.18) involve the dependence in Δ and j_\pm only through $e^{-(\Delta-2)\beta}$ and $\sin\left(\frac{2j_+ + 1}{2} \alpha_+\right) \sin\left(\frac{2j_- + 1}{2} \alpha_-\right)$ which are nothing but the numerator of the character $\chi_{\Delta, [j_+, j_-]}$ (5.15). Therefore, similarly to AdS_4 case, this observation allows to relate the zeta function $\zeta_{\Delta, [j_+, j_-]}$ to the character $\chi_{\Delta, [j_+, j_-]}$. An important advantage from the AdS_4 case — where $\tanh \pi u$ term (4.22) complicates the integral — is that the original u -integral in (5.10) can be exactly evaluated in AdS_5 as

$$\sqrt{\pi} \int_0^\infty du u^{2n} \left(\frac{\beta}{2u} \right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta u) = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(z - n)} \left(\frac{\beta}{2} \right)^{2(z-1-n)} [z > n], \quad (5.19)$$

after exchanging the order of integrals in β and u . Finally, we obtain the zeta function as the sum of three pieces:

$$\zeta_{\mathcal{H}}(z) := \zeta_{\mathcal{H}|2}(z) + \zeta_{\mathcal{H}|1}(z) + \zeta_{\mathcal{H}|0}(z), \quad (5.20)$$

where $\zeta_{\mathcal{H}|n}$ are the Mellin transforms,

$$\frac{\Gamma(z) \zeta_{\mathcal{H}|n}(z)}{\log R} = \int_0^\infty d\beta \frac{\left(\frac{\beta}{2}\right)^{2(z-1-n)}}{\Gamma(z-n)} f_{\mathcal{H}|n}(\beta), \quad (5.21)$$

of the functions $f_{\mathcal{H}|n}(\beta)$ given by

$$f_{\mathcal{H}|2}(\beta) = \frac{1}{8} \partial_{\alpha_+} \partial_{\alpha_-} \left[\eta(\beta, \alpha_+, \alpha_-) \chi_{\mathcal{H}}(\beta, \alpha_+, \alpha_-) \right]_{\alpha_\pm=0} = \frac{\sinh^4 \frac{\beta}{2}}{2} \chi_{\mathcal{H}}(\beta, 0, 0), \quad (5.22)$$

$$f_{\mathcal{H}|1}(\beta) = -\frac{1}{6} \partial_{\alpha_+} \partial_{\alpha_-} \left(\partial_{\alpha_+}^2 + \partial_{\alpha_-}^2 \right) \left[\eta(\beta, \alpha_+, \alpha_-) \chi_{\mathcal{H}}(\beta, \alpha_+, \alpha_-) \right]_{\alpha_\pm=0} \\ = \sinh^2 \frac{\beta}{2} \left[\frac{\sinh^2 \frac{\beta}{2}}{3} - 1 - \sinh^2 \frac{\beta}{2} (\partial_{\alpha_1}^2 + \partial_{\alpha_2}^2) \right] \chi_{\mathcal{H}}(\beta, \alpha_1, \alpha_2) \Big|_{\alpha_i=0}, \quad (5.23)$$

and

$$f_{\mathcal{H}|0}(\beta) = \frac{1}{6} \partial_{\alpha_+} \partial_{\alpha_-} \left(\partial_{\alpha_+}^2 - \partial_{\alpha_-}^2 \right)^2 \left[\eta(\beta, \alpha_+, \alpha_-) \chi_{\mathcal{H}}(\beta, \alpha_+, \alpha_-) \right]_{\alpha_\pm=0} \\ = \left[1 + \frac{\sinh^2 \frac{\beta}{2} (3 - \sinh^2 \frac{\beta}{2})}{3} (\partial_{\alpha_1}^2 + \partial_{\alpha_2}^2) \right. \\ \left. - \frac{\sinh^4 \frac{\beta}{2}}{3} (\partial_{\alpha_1}^4 - 12 \partial_{\alpha_1}^2 \partial_{\alpha_2}^2 + \partial_{\alpha_2}^4) \right] \chi_{\mathcal{H}}(\beta, \alpha_1, \alpha_2) \Big|_{\alpha_i=0}. \quad (5.24)$$

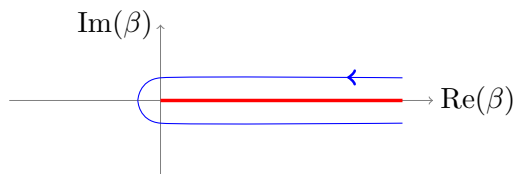


Figure 2. Integration contour for the zeta function.

Note that we have related the zeta function of a set of fields given by a Hilbert space \mathcal{H} ,

$$\zeta_{\mathcal{H}}(z) = \sum_{\Delta, j_+, j_-} N_{\Delta, [j_+, j_-]}^{\mathcal{H}} \zeta_{\Delta, [j_+, j_-]}(z), \quad (5.25)$$

to the corresponding character,

$$\chi_{\mathcal{H}}(\beta, \alpha_+, \alpha_-) = \sum_{\Delta, j_+, j_-} N_{\Delta, [j_+, j_-]}^{\mathcal{H}} \chi_{\Delta, [j_+, j_-]}(\beta, \alpha_+, \alpha_-), \quad (5.26)$$

where $N_{\Delta, [j_+, j_-]}^{\mathcal{H}}$ is the multiplicity of $\mathcal{V}(\Delta, [j_+, j_-])$ representation in the space \mathcal{H} . The existence of such relation is due to the fact that the formulas are linear and do not involve any explicit dependence on Δ or j_{\pm} .

Let us emphasize that in AdS_5 it was not necessary to change the regularization scheme from the ordinary zeta function $\zeta_{\mathcal{H}}$ to a deformed one $\tilde{\zeta}_{\mathcal{H}}$ as in AdS_4 . Hence, what we shall compute in the following are the standard zeta functions. This should be the case also for other odd dimensional AdS spaces where the absence of $\tanh \pi u$ term makes possible to evaluate the u integral.

Another important property of AdS_5 zeta function, which should hold in other odd dimensions, is the presence of gamma functions in the right hand sides of the formula (5.21). Thanks to this property, one can easily show that the UV divergence of the vacuum energy — corresponding to $\zeta_{\mathcal{H}}(0)$ — is universally absent, as is a well-known fact of odd dimensions:

$$\int_0^{\infty} d\beta \frac{\left(\frac{\beta}{2}\right)^{2(z-1-n)}}{\Gamma(z-n)} f_{\mathcal{H}|n}(\beta) = -2\gamma_{\mathcal{H}|n} + \mathcal{O}(z). \quad (5.27)$$

Moreover the finite part of the vacuum energy will be entirely captured by the divergence arising from the neighborhood of $\beta = 0$. If the function $f_{\mathcal{H}|n}$ does not have any singularity around the positive real axis of β except for the pole at $\beta = 0$, then the integral (5.27) with a sufficient large $\text{Re}(z)$ can be recast into to the contour integral,

$$\frac{i}{2 \sin(2\pi z)} \oint_C d\beta \frac{\left(-\frac{\beta}{2}\right)^{2(z-1-n)}}{\Gamma(z-n)} f_{\mathcal{H}|n}(\beta), \quad (5.28)$$

where the contour encircles the branch cut generated by $\beta^{2(z-1-n)}$ in the counter-clockwise direction (see figure 2).

Differently from the form (5.27), the above contour integral is well-defined for any value of z , hence we can directly put $z = 0$. With this evaluation, the integrand becomes

free from branch cut and the contour can be shrunk to a small circle around $\beta = 0$. In the end, the finite part $\gamma_{\mathcal{H}|n}$ will be given by the residue,

$$\gamma_{\mathcal{H}|n} = -(-4)^n n! \oint \frac{d\beta}{2\pi i} \frac{f_{\mathcal{H}|n}(\beta)}{\beta^{2(n+1)}}. \quad (5.29)$$

Practically, this amounts to expanding the functions $f_{\mathcal{H}|2}$, $f_{\mathcal{H}|1}$ and $f_{\mathcal{H}|0}$ around $\beta = 0$ and picking up the term proportional to β^5 , β^3 and β , respectively. Note that the residue vanishes for any even functions $f_{\mathcal{H}|n}(\beta)$, which is guarantied when the character $\chi_{\mathcal{H}}$ itself is even in β . In the end, the one-loop vacuum energy of the AdS₅ theory with the spectrum \mathcal{H} is given by the sum of three coefficients as

$$\Gamma_{\mathcal{H}}^{(1)\text{ren}} = \log R (\gamma_{\mathcal{H}|2} + \gamma_{\mathcal{H}|1} + \gamma_{\mathcal{H}|0}). \quad (5.30)$$

To summarize, the one-loop vacuum energy can be obtained from

Prescription 1: extracting β^{2n+1} coefficients of the functions $f_{\mathcal{H}|n}$

Prescription 2: the integral (5.27) with an analytic continuation on z

Prescription 3: the contour integral (5.28)

These three prescriptions are equivalent as far as the functions $f_{\mathcal{H}|n}$ free from any singularity around positive real β axis except for the pole at $\beta = 0$. This is the case for any one particle state in AdS₅, as well as for the spectrum of Vasiliev's theory. However, we will see that it is no more true for the AdS dual to a matrix model CFT. We shall come back to this issue after considering a few examples: the vector models as the first example, then the sample calculations for the spectra of the second and third Regge trajectories (that correspond to the CFT operators involving three and four fields ϕ , respectively). For a better illustration, we will consider both the exact evaluation of the β integral keeping the z dependence and the residue results (5.29) and (5.30).

5.3 Test with vector models

In order to test the result in the previous section, we compute the zeta function of the non-minimal/minimal Vasiliev theory, dual to free $U(N)/O(N)$ model CFT.

Let us first consider the non-minimal Vasiliev theory, whose spectrum corresponds to the tensor product of two Rac representation. The character of Rac obtained in (2.50) can be written in terms of $(\beta, \alpha_1, \alpha_2)$ variables as

$$\chi_{\text{Rac}}(\beta, \alpha_1, \alpha_2) = \frac{\sinh \beta}{2 (\cosh \beta - \cos \alpha_1) (\cosh \beta - \cos \alpha_2)}, \quad (5.31)$$

hence the character of the non-minimal model is

$$\chi_{\text{non-min}}(\beta, \alpha_1, \alpha_2) = [\chi_{\text{Rac}}(\beta, \alpha_1, \alpha_2)]^2 = \frac{\sinh^2 \beta}{4 (\cosh \beta - \cos \alpha_1)^2 (\cosh \beta - \cos \alpha_2)^2}. \quad (5.32)$$

One can first notice that the above character is even in β hence the residues (5.29) vanish implying that the one-loop vacuum energy vanishes. We may nevertheless proceed to

evaluate the zeta function for more concrete understanding. For the evaluation of the zeta function, we need to first calculate the corresponding $f_{\text{non-min}|n}(\beta)$ functions. They are given by

$$f_{\text{non-min}|2}(\beta) = \frac{\coth^2 \frac{\beta}{2}}{32}, \quad f_{\text{non-min}|1}(\beta) = \frac{\coth^2 \frac{\beta}{2}}{48} - \frac{\coth^2 \frac{\beta}{2}}{24 \sinh^2 \frac{\beta}{2}}, \quad f_{\text{non-min}|0}(\beta) = 0. \quad (5.33)$$

With these we can obtain $\zeta_{\text{non-min}|n}(z)$'s by performing Mellin transforms as (5.21). However, the integral is divergent for large β for a z which regularizes the small β divergence. Actually this large β divergence is due to the contribution of $\mathcal{D}(2,0)$ which is the lightest scalar field in AdS_5 in the spectrum. The latter has already ill-defined zeta function at the level of (5.10). One can regularize this divergence by increasing its mass by infinitesimally small amount as in [7]. This can be realized by replacing the Δ value from 2 to $2 + \epsilon$, and it amounts to inserting a $e^{-\epsilon\beta}$ term in the β integral. In this scheme, we get

$$\begin{aligned} & \frac{\Gamma(z) \zeta_{\text{non-min}}(z)}{\log R} \\ &= \lim_{\epsilon \rightarrow 0} \int_0^\infty d\beta e^{-\epsilon\beta} \left[\frac{\left(\frac{\beta}{2}\right)^{2(z-3)}}{\Gamma(z-2)} \frac{\coth^2 \frac{\beta}{2}}{32} + \frac{\left(\frac{\beta}{2}\right)^{2(z-2)}}{\Gamma(z-1)} \left(\frac{\coth^2 \frac{\beta}{2}}{48} - \frac{\coth^2 \frac{\beta}{2}}{24 \sinh^2 \frac{\beta}{2}} \right) \right] \\ &= \frac{9\Gamma(z - \frac{5}{2})\zeta(2z-6) - 8\Gamma(z - \frac{3}{2})\zeta(2z-6) + 2\Gamma(z - \frac{3}{2})\zeta(2z-4)}{72\sqrt{\pi}} \end{aligned} \quad (5.34)$$

where we used the integration method of appendix C. In the $z \rightarrow 0$ limit, the ratios of gamma functions in the right hand side are finite whereas the zeta functions vanish: $\zeta(-6) = \zeta(-4) = 0$. Hence, we can verify that the one-loop vacuum energy vanishes. Alternatively, we can use Laurent expansion of $f_{\text{non-min}|n}$ with the variable $q \equiv e^{-\beta}$. By this, we can check that the quantities $\gamma_{\text{non-min}|n}$ defined in (5.29) individually vanish for $n = 0, 1, 2$ (since the functions are even in β). This result again verify the vanishing of the one-loop vacuum energy.

We next consider the minimal Vasiliev theory with only even spins. The corresponding character is given again in terms of χ_{Rac} as (4.28). Since the $\chi_{\text{Rac}}^2 = \chi_{\text{non-min}}$ gives trivial vacuum energy, the only non-trivial contribution may come from the second term $\chi_{\text{Rac}}(2\beta, 2\alpha_1, 2\alpha_2)/2 = \chi_{\text{min}}(\beta, \alpha_1, \alpha_2) - \chi_{\text{non-min}}(\beta, \alpha_1, \alpha_2)/2 =: \chi_{\text{R}}(\beta, \alpha_1, \alpha_2)$. After some manipulation, one get

$$f_{\text{R}|2}(\beta) = \frac{\cosh \beta \sinh^4 \frac{\beta}{2}}{16 \sinh^3 \beta}, \quad f_{\text{R}|1}(\beta) = \frac{\cosh \beta (\sinh^2 \frac{\beta}{2} - 2) \sinh^6 \frac{\beta}{2}}{6 \sinh^5 \beta}, \quad f_{\text{R}|0}(\beta) = 0. \quad (5.35)$$

The corresponding zeta function is given by

$$\begin{aligned} & \frac{\Gamma(z) \zeta_{\text{R}}(z)}{\log R} \\ &= \frac{\Gamma(z)}{\log R} \left(\zeta_{\text{min}}(z) - \frac{1}{2} \zeta_{\text{non-min}}(z) \right) \\ &= \lim_{\epsilon \rightarrow 0} \int_0^\infty d\beta e^{-\epsilon\beta} \left[\frac{\left(\frac{\beta}{2}\right)^{2(z-3)}}{\Gamma(z-2)} \frac{\cosh \beta \sinh^4 \frac{\beta}{2}}{16 \sinh^3 \beta} + \frac{\left(\frac{\beta}{2}\right)^{2(z-2)}}{\Gamma(z-1)} \frac{\cosh \beta (\sinh^2 \frac{\beta}{2} - 2) \sinh^6 \frac{\beta}{2}}{6 \sinh^5 \beta} \right] \end{aligned}$$

$$\begin{aligned}
 &= -\frac{2^{1-4z} \Gamma(2z-5) [(2^{2z}-256) \zeta(2z-7) + (2^{2z}-64) \zeta(2z-5)]}{\Gamma(z-2)} \\
 &\quad - \frac{2^{-4z} \Gamma(2z-3) [(2^{2z}-256) \zeta(2z-7) + 3(2^{1+2z}-128) \zeta(2z-5) + (2^{2z}-16) \zeta(2z-3)]}{3\Gamma(z-1)} \\
 &= -\frac{1}{45} + \mathcal{O}(z).
 \end{aligned} \tag{5.36}$$

We therefore correctly reproduce the result of [7] for the one-loop vacuum energy of the minimal Vasiliev theory

$$\Gamma_{\min}^{(1)\text{ren}} = -\frac{1}{2} \zeta'_R(0) = \frac{\log R}{90}. \tag{5.37}$$

Again, instead of computing the zeta function explicitly, one can directly identify the vacuum energy from the residue calculations. By Laurent expanding the $f_{R|n}(\beta)$ we get

$$\gamma_{R|2} = \frac{13}{1920}, \quad \gamma_{R|1} = \frac{5}{1152}, \quad \gamma_{R|0} = 0, \tag{5.38}$$

whose sum again gives $1/90$. Therefore, the examples of vector models show the agreement. We will now compare the above result with $\Gamma_{\text{Rac}}^{(1)\text{ren}}$, the one-loop vacuum energy associated to the Rac representation in AdS_5 . This does not represent a propagating degree of freedom in the bulk. Nonetheless, we can formally define and evaluate a one-loop determinant corresponding to this field. We will find that the answer correctly reproduces the a -anomaly of the conformal scalar on the boundary. See [13] for related discussions. We also remark here that this result will be useful for computing the one-loop vacuum energy of the AdS dual of the free $\text{SU}(N)$ adjoint scalar field theory as well. Again, we first calculate the functions $f_{\text{Rac}|n}$ and get

$$f_{\text{Rac}|2}(\beta) = \frac{\sinh \beta}{16}, \quad f_{\text{Rac}|1}(\beta) = \frac{\sinh \beta}{24}, \quad f_{\text{Rac}|0}(\beta) = 0. \tag{5.39}$$

With these, the zeta function is given by the Mellin transform,

$$\begin{aligned}
 \frac{\Gamma(z) \zeta_{\text{Rac}}(z)}{\log R} &= \lim_{\epsilon \rightarrow 0} \int_0^\infty d\beta e^{-\epsilon \beta} \left[\frac{\left(\frac{\beta}{2}\right)^{2(z-3)} \sinh \beta}{\Gamma(z-2) 16} + \frac{\left(\frac{\beta}{2}\right)^{2(z-2)} \sinh \beta}{\Gamma(z-1) 24} \right] \\
 &= -\frac{(-1)^{2z} [1 + (-1)^{2z}] (z-1) \Gamma(z - \frac{5}{2})}{48 \sqrt{\pi}} = -\frac{1}{45} + \mathcal{O}(z),
 \end{aligned} \tag{5.40}$$

where we have again regularized the divergence coming from large β region by introducing a $e^{-\epsilon \beta}$ damping factor. This divergence arises because the conformal weight is smaller than 2. By expanding the zeta function around $z = 0$, one can show that it gives the same value as the vacuum energy of the conformal scalar on the boundary. Once again, the finite part of the vacuum energy can be directly extracted by expanding (5.39) around $\beta = 0$, and we immediately get

$$\gamma_{\text{Rac}|2} = -\frac{1}{60}, \quad \gamma_{\text{Rac}|1} = \frac{1}{36}, \quad \gamma_{\text{Rac}|0} = 0. \tag{5.41}$$

By summing the above three contributions, we get

$$\Gamma_{\text{Rac}}^{(1)\text{ren}} = \frac{\log R}{90}. \tag{5.42}$$

Interestingly again, as mentioned at the outset, the IR log divergence of ζ_{Rac} gives the UV log divergence of the vacuum energy of the conformal scalar on the boundary.

5.4 Vacuum energy for the AdS dual of $SU(N)$ adjoint model

We now turn to the main result of the paper, one-loop vacuum energy computation of the AdS_5 theory which is dual to the free $SU(N)$ adjoint scalar CFT on S^4 . We first compute the vacuum energy of the fields in the first few Regge trajectories using the method of residues, following the prescription (5.29). We present a trend of the vacuum energy growth by the power of the fields ϕ , exhibited in figure 3. Next, we will take the limit $N \rightarrow \infty$, and compute the one-loop vacuum energy for the corresponding theory. This indicates a non-trivial shift in the relation between the bulk dimensionless coupling g and the boundary parameter N .

5.4.1 Vacuum energies for a few low orders

We first carry out the determination of the one-loop vacuum energy of the fields in the first few Regge trajectories using the method of residues. We have explicitly evaluated and exhibited the contributions of terms up to order 4 in the fields ϕ and results up to order 32 are then displayed graphically. While the overall pattern for the one-loop vacuum energy is chaotic, and indeed non-monotonic as well, we do observe the trend that the vacuum energies corresponding to $n = 2, 4, 8, 16, 32$ are exactly 1, 2, 4, 8, 16 times of the $n = 1$ case, respectively. We start with the order two contribution.

Order two. The order two spectrum coincides with that of minimal Vasiliev theory, hence the vacuum energy is equal to that of Rac:

$$\Gamma_{\text{cyc}^2}^{(1)\text{ren}} = \frac{\log R}{90} \simeq 0.0111111 \log R. \quad (5.43)$$

Order three. The order three cyclic character is given analogously to (4.36). From that, we first calculate $f_{\text{cyc}^3|n}(\beta)$'s and their residues,

$$\gamma_{\text{cyc}^3|2} = \frac{7627}{475200}, \quad \gamma_{\text{cyc}^3|1} = \frac{46483}{3991680}, \quad \gamma_{\text{cyc}^3|0} = -\frac{3071}{561330}. \quad (5.44)$$

By summing the above, we get

$$\Gamma_{\text{cyc}^3}^{(1)\text{ren}} = \frac{362911}{16329600} \log R \simeq 0.0222241 \log R, \quad (5.45)$$

which is roughly twice of the order two contribution.

Order four. The order four cyclic character is given analogously to (4.39), and we obtain similarly the residues,

$$\gamma_{\text{cyc}^4|2} = \frac{253}{15360}, \quad \gamma_{\text{cyc}^4|1} = \frac{125}{9216}, \quad \gamma_{\text{cyc}^4|0} = -\frac{1}{128}. \quad (5.46)$$

whose sum simplifies ending up with

$$\Gamma_{\text{cyc}^3}^{(1)\text{ren}} = \frac{\log R}{45} \simeq 0.0222222 \log R. \quad (5.47)$$

Interestingly, the order four contribution is exactly twice of the contribution of Rac or the order two. We can see that the vacuum energies do not increase monotonically as the order four part is slightly smaller than the order three one.

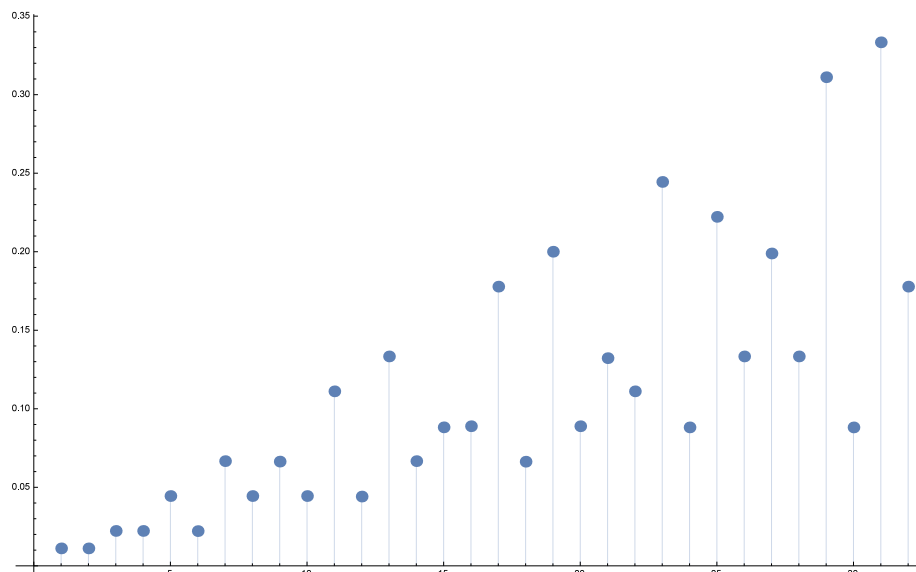


Figure 3. Vacuum energies of AdS_5 fields in different orders.

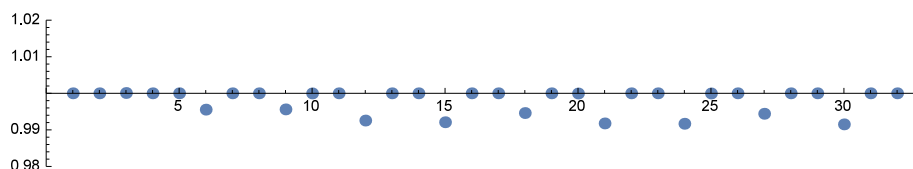


Figure 4. Plot of $\Gamma_{\text{cyc}^n}^{(1)\text{ren}} / (\varphi(n) \Gamma_{\text{Rac}}^{(1)\text{ren}})$ in AdS_5 from $n = 1$ to 32.

Higher orders. In order to have a better idea, we can proceed to calculate the higher order contributions to the vacuum energy. Figure 3 shows the values of the vacuum energies for the fields corresponding to χ_{cyc^n} up to the order $n = 32$. As we have commented previously, the energies show an approximate linear growth with some chaotic oscillations. Additionally, the vacuum energies corresponding to $n = 2, 4, 8, 16, 32$ are exactly 1, 2, 4, 8, 16 times of the $n = 1$ case, respectively. One can notice that the first eight values of the vacuum energies show very similar pattern as the AdS_4 case (figure 1). In fact, both vacuum energies follow the pattern of the Euler totient function very closely. See figure 4. Apart from the $n = 2^m$ cases, the values plotted in Fig 4 are not exactly 1 but have very small fluctuations. It would be interesting to investigate this pattern further.

5.4.2 Vacuum energy in different slices

When the rank of the gauge group becomes infinite, the character encoding the set of all single trace operators in the CFT can be simplified to obtain an expression in terms of the Euler totient function, given in the second equality of (2.84), reproduced here for convenience:

$$\chi_{\text{adj}} = -\chi_{\text{Rac}} + \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \chi_{\log, k}. \quad (5.48)$$

The $\chi_{\log,k}$ are given in the AdS₅ case by

$$\chi_{\log,k}(\beta, \alpha_1, \alpha_2) = -\log[1 - \chi_{\text{Rac}}(k\beta, k\alpha_1, k\alpha_2)] . \quad (5.49)$$

The first term in (5.48) subtracts the single field contribution from the rest, and its contribution to the vacuum energy has been calculated in (5.42). We will therefore focus on the logarithmic term (5.49) which is new. As we already argued in the AdS₄ case, carrying out this computation while summing over all k leads to an infinite number of singular points in the β plane, clustering around $\beta = 0$, making the partition function highly non-analytic. Therefore to carry out this computation in a well-defined way, we need to work at a fixed k , compute the one-loop vacuum energy contribution, and then sum over all different k contributions with the weight $\varphi(k)/k$. For these reasons, from now on let us focus on $\chi_{\log,k}$ (5.49) and calculate the corresponding contribution to the vacuum energy $\Gamma_{\log,k}^{(1)\text{ren}}$. For that, we need to first identify the functions $f_{\log,k|n}(\beta)$. After some computations, we get

$$\begin{aligned} f_{\log,k|2}(\beta) &= \frac{\sinh^4 \frac{\beta}{2}}{2} \log \left[1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}} \right] , \\ f_{\log,k|1}(\beta) &= \frac{\sinh^2 \frac{\beta}{2} \left(\sinh^2 \frac{\beta}{2} - 3 \right)}{6} \log \left[1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}} \right] + \frac{k^2 \sinh^4 \frac{\beta}{2} \coth \frac{k\beta}{2}}{\sinh(k\beta) - 8 \sinh^4 \frac{k\beta}{2}} , \\ f_{\log,k|0}(\beta) &= \log \left[1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}} \right] - \frac{2}{3} k^2 \left[(k^2 - 1) \sinh^2 \frac{\beta}{2} + 3 \right] \frac{\sinh^2 \frac{\beta}{2} \coth \frac{k\beta}{2}}{\sinh(k\beta) - 8 \sinh^4 \frac{k\beta}{2}} \\ &\quad - 2 k^4 \left(\frac{\sinh^2 \frac{\beta}{2} \coth \frac{k\beta}{2}}{\sinh(k\beta) - 8 \sinh^4 \frac{k\beta}{2}} \right)^2 . \end{aligned} \quad (5.50)$$

The next step is the identification of $\gamma_{\log,k|n}$ with (5.29). Note that the issue of different prescriptions enters here. As one can see, the function $f_{\log,k|n}(\beta)$ do have additional branch cuts on the positive real axis of β . In fact, such singularities precisely coincide with the Hagedorn phase transition which appears at the thermal AdS partition function, where we interpret the integration variable β as the inverse temperature. In our work, we are focusing on the AdS space with sphere boundary hence there is no notion of temperature a priori. Nevertheless, the technical simplification of the vacuum energy calls for the use of the full character. This can be physically interpreted as the generalized partition function, which does see the Hagedorn transition at the specific point $\beta = \beta_H$. We emphasize again the β does not carry a thermal/geometric meaning on the boundary/bulk of the AdS space, but simply enters to address properly the spectrum of the theory in consideration. As we have shown before, the one-loop vacuum energy is given through the integral of such partition function and the singularity arising at β_H introduces an ambiguity of the prescription. In other words, as we have seen from the computation of the vacuum energy of a fixed Regge trajectory, the vacuum energy increases as the order of ϕ in the dual CFT operator increases. Hence, the full one-loop vacuum energy will be given as a divergent series. This divergence, arising while summing over infinitely many trajectories, is not automatically regularized by the introduction of the UV regulator z as in the Vasiliev's model case, but

requires a new regularization prescription. This shows a clear difference between Vasiliev's theory (dual to vector models) and stringy AdS theory (dual to matrix models).

The necessity of introducing a new regularization method is translated here to the choice of prescription among three possibilities proposed in section 5.2. Let us examine the three prescriptions, one by one, for the AdS theory dual to the $SU(N)$ adjoint model.

Prescription 1. The prescription 1 is the simplest option and has well defined meaning even for the functions $f_{\log,k|n}(\beta)$: it is sufficient to expand the functions $f_{\log,k|n}(\beta)$ around the $\beta = 0$ point. Considering first $f_{\log,k|2}(\beta)$, one can show that the β^5 coefficient is absent:

$$f_{\log,k|2}(\beta) = \frac{-3 \log \beta + \log(-\frac{2}{k^3})}{32} \beta^4 + \frac{-3 \log \beta + \log(-\frac{2}{k^3})}{192} \beta^6 - \frac{k^3}{64} \beta^7 + \mathcal{O}(\beta^8), \quad (5.51)$$

implying $\gamma_{\log,k|2} = 0$. Similarly, we obtain the expansions of the remaining functions as

$$f_{\log,k|1}(\beta) = \frac{3 \log \beta - \log(-\frac{2}{k^3}) + 1}{8} \beta^2 - \frac{k^2 - 2}{96} \beta^4 + \frac{k^3}{8} \beta^5 + \mathcal{O}(\beta^6), \quad (5.52)$$

$$f_{\log,k|0}(\beta) = -3 \log \beta + \log\left(-\frac{2}{k^3}\right) - \frac{3}{2} + \frac{k^2 - 1}{12} \beta^2 - \frac{3k^2}{2} \beta^3 + \mathcal{O}(\beta^4), \quad (5.53)$$

and can verify that they miss the β^3 and β coefficients, respectively. Hence, $\gamma_{\log,k|1} = \gamma_{\log,k|0} = 0$, and in this prescription, we are led to conclude that the one-loop vacuum energy for a fixed slice vanishes.

Prescription 2. The prescription 2 is the integral along the positive real axis of β . This integral is convergent only for large enough z hence requires an analytic continuation on z . In the following, we sketch how we can carry out the computation of such integral for $\zeta_{\log,k|2}(z)$. We first recast the integral into

$$\begin{aligned} \frac{\Gamma(z) \zeta_{\log,k|2}(z)}{\log R} &= -\frac{(2k)^{5-2z} \Gamma(2z-5)}{16 \Gamma(z-2)} \times \\ &\times [\Psi_k(p_1, 2z-5) + \Psi_k(p_2, 2z-5) + \Psi_k(p_3, 2z-5) - 3 \Psi_k(1, 2z-5)], \end{aligned} \quad (5.54)$$

by introducing

$$\Psi_k(p, z) = 6 \tilde{\Phi}(p, z, 0) - 4 \tilde{\Phi}\left(p, z, -\frac{1}{k}\right) - 4 \tilde{\Phi}\left(p, z, \frac{1}{k}\right) + \tilde{\Phi}\left(p, z, -\frac{2}{k}\right) + \tilde{\Phi}\left(p, z, \frac{2}{k}\right). \quad (5.55)$$

Here $\tilde{\Phi}(p, z, a)$ is the function defined for the region $|p| < 1$ by

$$\tilde{\Phi}(p, z, a) = \int_0^\infty d\beta \frac{\beta^{z-1}}{\Gamma(z)} \log(1 - p e^{-\beta}) e^{-a\beta}, \quad (5.56)$$

and for the other region by analytic continuations. The p_1 , p_2 and p_3 are defined through the factorization,¹⁹

$$1 - 4q + 2q^2 - q^3 = (1 - p_1 q)(1 - p_2 q)(1 - p_3 q). \quad (5.57)$$

¹⁹The argument of the log function which appears in all of the $f_{\log,k|n}$ is $1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}} = \frac{1-4q^k+2q^{2k}-q^{3k}}{(1-q^k)^3}$, where $q = e^{-\beta}$. Factorizing the numerator as in eq. (5.57) enables us to replace $\log\left[1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}}\right]$ by $\log(1 - p_1 q^k) + \log(1 - p_2 q^k) + \log(1 - p_3 q^k) - 3 \log(1 - q^k)$, from which we obtain decompositions like eq. (5.54). See appendix C for details.

For $\text{Re}(z) > 1$ and $p \neq 1$, the integral (5.56) is equivalent to the contour integral,

$$\tilde{\Phi}(p, z, a) = \frac{i}{2 \sin \pi z} \oint d\beta \frac{(-\beta)^{z-1}}{\Gamma(z)} \log(1 - p e^{-\beta}) e^{-a\beta}, \quad (5.58)$$

with the contour depicted in figure 2. The above is again well-defined for any value of z , hence one can immediately put $z = -5$ and consider the residue at the origin. This way, we obtain

$$\Psi_k(p, -5) = \frac{120}{k^4} \frac{p}{1-p}. \quad (5.59)$$

Note that the above result can be applied to $p = p_1, p_2$ and p_3 but not to $p = 1$, as one can check that (5.59) diverges in the latter case. This means that Prescription 2 does not give a finite value for $\zeta'_{\log, k|2}(0)$ even after an analytic continuation on z . To proceed, we need to properly extract a finite part from $\Psi_k(1, -5)$. The shift of the branch point at the origin to $\beta = -\epsilon$ corresponds to the replacement of $\Psi_k(1, -5)$ by

$$\Psi_k(e^{-\epsilon}, -5) = \frac{120}{k^4} \left(\frac{1}{\epsilon} - \frac{1}{2} + \mathcal{O}(\epsilon) \right). \quad (5.60)$$

By taking only the finite part $-1/2$, the total contribution for (5.54) is proportional to

$$\frac{p_1}{1-p_1} + \frac{p_2}{1-p_2} + \frac{p_3}{1-p_3} - 3 \left(-\frac{1}{2} \right). \quad (5.61)$$

Using (5.57) one can show that the above four terms exactly cancel each others implying $\zeta'_{\log, k|2}(0) = 0$. Let us remark that the regularization (5.60) is equivalent to taking only the β^5 term from the integrand of (5.56) for the residue and ignoring the presence of the branch cut. Therefore, Prescription 2 with the regularization (5.60) is equivalent to Prescription 1. For $\zeta_{\log, k|1}$ and $\zeta_{\log, k|0}$, one can do similar analysis and show their first derivatives vanish at $z = 0$. See appendix C for more details.

Let us focus on the result obtained in Prescription 1 and 2 and provide some interpretation. As all three contributions $\gamma_{\log, k|2}$, $\gamma_{\log, k|1}$ and $\gamma_{\log, k|0}$ vanish. Hence, we can conclude that the k -th contribution to the vacuum energy vanishes according to (5.30):

$$\Gamma_{\log, k}^{(1) \text{ ren}} = 0. \quad (5.62)$$

Because each k contribution vanishes, the total vacuum energy is

$$\Gamma_{\text{AdS}_5}^{(1) \text{ ren}} = -\Gamma_{\text{Rac}}^{(1) \text{ ren}} + \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \Gamma_{\log, k}^{(1) \text{ ren}} = -\Gamma_{\text{Rac}}^{(1) \text{ ren}}. \quad (5.63)$$

Therefore, we conclude that the expansion of free energy near AdS_5 vacuum is

$$\Gamma_{\text{AdS}_5}^{(1) \text{ ren}} = \left(g \mathcal{L}_0 - \frac{1}{90} \right) \log R + \mathcal{O}(g^{-1}), \quad (5.64)$$

where $\mathcal{L}_0 = S_0/\text{Vol}_{\text{AdS}_5}$ is the classical Lagrangian evaluated on the AdS_5 vacuum solution (see (3.13)), and g is the dimensionless coupling constant defined in (3.9). From the

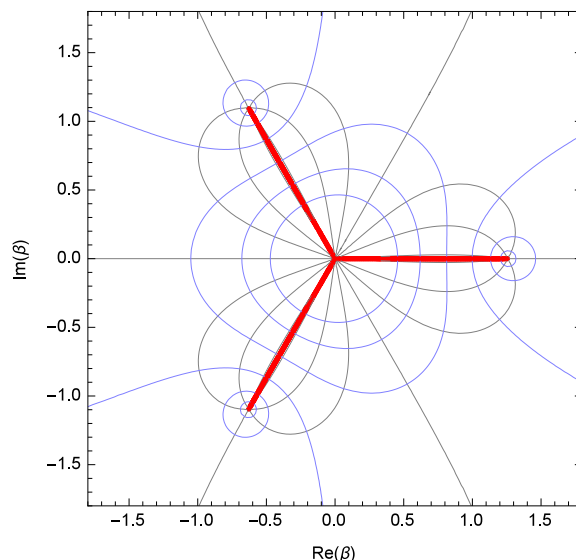


Figure 5. The blue/grey lines are the contour plots for the real/imaginary part of the function (5.68). The red line corresponds to the branch cut.

AdS/CFT correspondence, the free energy F_{CFT_4} of boundary scalar field theory should be identified to the Γ_{AdS_5} . The free energy of conformal $\text{SU}(N)$ matrix scalar on the S^4 has a logarithmic divergence corresponding to the conformal a -anomaly [97, 98],

$$F_{\text{CFT}_4} = \frac{N^2 - 1}{90} \log \Lambda + \mathcal{O}(\Lambda^0). \quad (5.65)$$

Using the correspondence between IR and UV divergences respectively in AdS_5 and S^4 , we get

$$g \mathcal{L}_0 - \frac{1}{90} = \frac{N^2 - 1}{90}. \quad (5.66)$$

As in the vector model cases [6–8], this formula suggests the relations,

$$g = N^2, \quad \mathcal{L}_0 = \frac{1}{90}. \quad (5.67)$$

It may be worth to note that the second equation is compatible with the vector model cases, assuming that the matter sectors do not contribute to the Lagrangian value \mathcal{L}_0 . This is a reasonable assumption from ordinary field theory point of view because only gravity can have a non-trivial background value.

Prescription 3. Finally in the prescription 3, we have to consider a contour which encircles all the singularities of the integrand. In particular, the logarithm function,

$$\log \left[1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}} \right], \quad (5.68)$$

generates the branch cuts which is depicted in figure 5. With the evaluation $z = 0$, the branch cut of $(-\beta)^{2(z-1-n)}$ along the positive real axis of β disappears, and the remaining

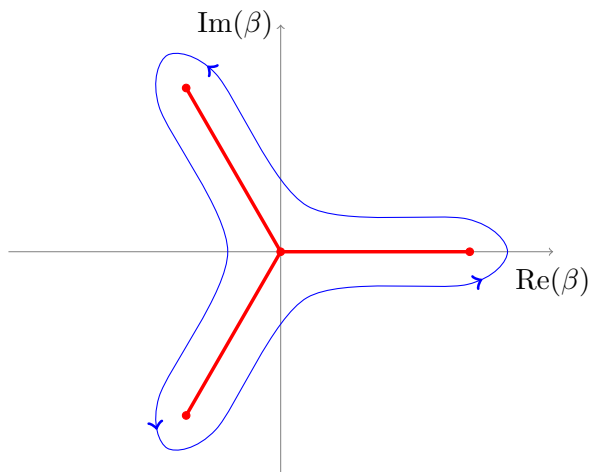


Figure 6. Integration contour for $\zeta'_{\log,k|n}(0)$.

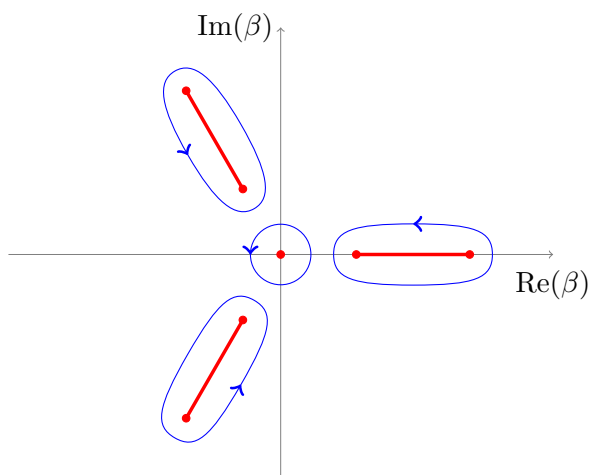


Figure 7. Integration contours for shifted branch points.

singularities are composed of the branch cuts in figure 5 and the poles at the end points of the cuts.

Hence we can consider the contour of figure 6. Such integral will give a finite and non-trivial result for a fixed slice k . Once the integral is evaluated, we can sum them for all k according to (5.48) to get the full vacuum energy. However, in this paper we do not evaluate them as we could not find any analytic measure to do so. Instead, in order to see the relation of Prescription 3 to the others, let us consider a deformation of the branch points at $\beta = 0$ while keeping all the poles untouched. The original integral can now be split into four pieces as in figure 7. After evaluating the four contour integrals separately, we can eventually take the limit that the shifted branch points tend back to the origin. Each of four contributions diverges in the latter limit, while their sum is finite. In this set-up, Prescription 1 would correspond to taking the finite part of the contour around the origin but discarding all other contributions.

6 Conclusion

Let us briefly summarize the objectives and the results obtained in this paper. The physics we attempted to explore is the holography of the free $SU(N)$ adjoint scalar CFT in the $N \rightarrow \infty$ limit. We assume this theory to admit an AdS dual, an extension of Vasiliev's theory with infinitely many higher spin massive multiplets. This theory can be considered as a toy model for more realistic and/or interesting stringy models. The field content of the AdS theory is to match with the spectrum of single trace operators in the CFT. This is a necessary condition for formulating a meaningful AdS/CFT duality in all known cases. The main aspect of this holography that we investigated is the one-loop correction of the AdS theory which ought to correspond to the first $1/N$ correction of the CFT. Since the CFT is free, the latter correction vanishes implying the triviality of the AdS one-loop effect.

The one-loop triviality of AdS theory is a highly non-trivial property as an infinite number of contributions from each field contents should sum up to cancel precisely. In order to test this property, we first analyzed the single trace operator content of the CFT by decomposing tensor product of the singleton representation which accounts the degrees of freedom of free conformal scalar. Explicit decompositions were carried out for the three and four tensor products in the three dimensional CFT, and in principle the analysis is extendable to other dimensions and higher powers as well. We found that the decomposition rules increase in complexity as higher powers of singletons are considered, but in every case explicit closed form expressions can be obtained.

Next, we turn to the simplest one-loop effect, the vacuum energy of the AdS theory. Using the explicit results obtained for the spectrum of single trace operators, we compute the vacuum energies of the AdS fields in a first few Regge trajectories. However, this method quickly becomes prohibitive both due to increasing complexity of the spectrum and the careful regularization required at various steps in the calculation.

We therefore developed a new formalism for writing down the zeta function, exploiting the fact that the character of the singleton encodes the spectrum of single trace operators. This formalism greatly simplifies computations of the one-loop vacuum energies. In the case of AdS_4 , it allowed us to calculate the vacuum energies for the first 8 trajectory fields. In the case of AdS_5 , the integral involved becomes even simpler, hence we could calculate the vacuum energies for the first 32 trajectories. In both cases, the calculation of vacuum energy for any fixed trajectory can be done in principle, but requires more computing powers for higher trajectories. An analytic expression of the vacuum energy for an arbitrary level is nevertheless unavailable. It is partly due to the fact that the corresponding character involves the number theoretic Euler totient function.

In order to avoid the difficulty related to the Euler totient function, we consider the different summation (or slice) for the full vacuum energy which is valid only in the $N \rightarrow \infty$ limit. At this point, it might be worth to remark that the integral and series that we are considering — $\sum_n \int d\beta$ or $\sum_k \int d\beta$ — is all the time ‘two-dimensional’. This is to be compared with the two-dimensional fundamental domain of torus appearing in the

string loop:

$$\sum_{\Delta, \ell} N_{\Delta, \ell} \left(\text{circle with } \Delta, \ell \text{ below it} \right) = \sum_{n \text{ or } k} \int d\beta (\dots) = \left(\text{annulus with } \Delta, \ell \text{ below it} \right). \quad (6.1)$$

With the new slice, the AdS_4 vacuum energy for a given slice is given by an integral, which could be evaluated only by a numerical method (see appendix B). Turning to AdS_5 case, we note first that the derivation of the vacuum energy from the character is no more unique for AdS duals of matrix model CFTs. We considered three prescriptions which become equivalent for a single particle state in AdS_5 , but differ from each other for the matrix model case.

In the first prescription, we find that the vacuum energy exactly vanishes for each slice. This result can be equally obtained in the second prescription upon introducing an addition regularization. Hence, the one-loop vacuum energy of the AdS_5 theory dual to the free $\text{SU}(N)$ adjoint scalar CFT on S^4 coincides with minus of the free energy of a boundary scalar. This is somewhat analogous to the duality between the minimal Vasiliev theory and the free $O(N)$ vector model, hence suggests that the loop-expansion parameter in the AdS_5 theory should be identified to N^2 rather than $N^2 - 1$, the dimension of $\text{SU}(N)$.

In the last prescription, the vacuum energy for a fixed slice is given by a contour integral surrounding the singularities. The latter contains not only poles but also branch cuts, which are generated by the logarithm arising as a result of summing over trajectories. These singularities are precisely where the Hagedorn phase transition takes place. The appearance of such singularities and the possibility of their relevance to one-loop vacuum energy is interesting but also intriguing, because the parameter β is not a physical temperature for the CFT in our case. Rather it is just a parameter which is being used to count the number of the spectrum of the theory, and in this sense one might not expect the Hagedorn transition to play a role in the vacuum energy. In this work, for the technical reason, we have not explore sufficiently the contributions coming from the Hagedorn-related singularities. We hope to revisit this issue in near future. One interesting direction in this respect would be the computation of the vacuum energy in the thermal AdS_5 with $S^3 \times S^1$ boundary, where the Hagedorn transition has a physical meaning. We hope to report soon about the latter computation. Let us also note that in [99, 100], related issues have been discussed from the boundary theory point of view: the vacuum energy of large N gauge theories on $S^3 \times S^1$ was shown to vanish with a particular choice of branch.

Finally, the results contained in this paper can be extended in many directions. The most interesting one would be to apply these tests to explore the duality of tensionless string theory to free CFTs, e.g. the duality between the Type IIB string theory on $\text{AdS}_5 \times S^5$ and the $\mathcal{N} = 4$ Super Yang-Mills (SYM) with the gauge group $\text{SU}(N)$. The planar ($N \rightarrow \infty$) and free ($g_{YM}^2 N \rightarrow 0$) limit of this theory should correspond in the bulk to the tensionless limit of Type IIB string theory. While this phase of string theory is still fairly poorly understood, one may use the boundary CFT data available to us as a means of getting some insight into the bulk physics [20, 21, 23–25]. In particular, one may expect to identify

the field content of the bulk theory from the operator spectrum of the dual CFT, and carry out the checks we have done in this paper.

Additionally, for technical reasons we mainly focused on free field theories whose operator spectrum is completely classified by taking tensor products of the singleton representation. However, our methods allow us in principle to compute the one-loop vacuum energy for the bulk dual of any theory whose planar operator spectrum is known. These include the Chern-Simons with Matter theories of [101, 102] and the interacting theories obtained by flowing to Wilson-Fisher fixed points by turning on double-trace deformations for the free theory, much as in the case of vector models. These, and related questions, are work in progress and we hope to report on them soon.

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A Oscillator analysis for tensor products

In order to find out the explicit formulas for $N_{(n_a, n_b)}^{(\epsilon_1, \dots, \epsilon_k)}$, let us recast the branching (2.79) in terms of the character:

$$\chi_{(n_a, n_b)}(\Sigma) = \sum_{\epsilon_1 = \pm 1} \cdots \sum_{\epsilon_p = \pm 1} N_{(n_a, n_b)}^{(\epsilon_1, \dots, \epsilon_k)} \chi_{(\epsilon_1, \dots, \epsilon_k)}(\Sigma), \quad (\text{A.1})$$

where $\chi_{(n_a, n_b)}$ is the $O(k)$ character in the (n_a, n_b) representation. $\Sigma = \sigma_1^{\delta_1} \cdots \sigma_k^{\delta_k}$ (with $\delta_i = 0, 1$) is an element of $\mathbb{Z}_2^{\otimes k} \subset O(k)$, and $\chi_{(\epsilon_1, \dots, \epsilon_k)}$ is its character in the $(\epsilon_1, \dots, \epsilon_k)$ representation:

$$\chi_{(\epsilon_1, \dots, \epsilon_k)}(\sigma_1^{\delta_1} \cdots \sigma_k^{\delta_k}) = \chi_{\epsilon_1}(\sigma_1^{\delta_1}) \cdots \chi_{\epsilon_k}(\sigma_k^{\delta_k}), \quad \chi_{\epsilon}(\sigma^{\delta}) = \epsilon^{\delta}. \quad (\text{A.2})$$

The multiplicities can be obtained using these properties as

$$\begin{aligned} N_{(n_a, n_b)}^{(\epsilon_1, \dots, \epsilon_k)} &= \frac{1}{|\mathbb{Z}_2^{\otimes k}|} \sum_{\Sigma \in \mathbb{Z}_2^{\otimes k}} \chi_{(\epsilon_1, \dots, \epsilon_k)}(\Sigma^{-1}) \chi_{(n_a, n_b)}(\Sigma) \\ &= \frac{1}{2^k} \sum_{\delta_1=0,1} \cdots \sum_{\delta_k=0,1} \epsilon_1^{\delta_1} \cdots \epsilon_k^{\delta_k} \chi_{(n_a, n_b)}(\sigma_1^{\delta_1} \cdots \sigma_k^{\delta_k}). \end{aligned} \quad (\text{A.3})$$

From the above formula, one can see that the multiplicity (that is the tensor product decomposition) depends on the number of Di's and Rac's but not their order:

$$N_{(n_a, n_b)}^{(\epsilon_1, \dots, \epsilon_k)} = N_{(n_a, n_b)}^{[l, k-l]}, \quad k - 2l = \epsilon_1 + \dots + \epsilon_k, \quad (\text{A.4})$$

here l and $k - l$ are respectively the number of Di's and Rac's. The multiplicity $N_{(n_a, n_b)}^{[l, k-l]}$ is simply given by the $O(k)$ character as

$$\begin{aligned} N_{(n_a, n_b)}^{[l, k-l]} &= \frac{1}{2^k} \sum_{\delta_1=0,1} \dots \sum_{\delta_k=0,1} (-1)^{\delta_1 + \dots + \delta_l} \chi_{(n_a, n_b)}(\sigma_1^{\delta_1} \dots \sigma_k^{\delta_k}) \\ &= \frac{1}{2^k} \sum_{j=0}^k c_k^{[l, k-l]} \chi_{(n_a, n_b)}(\sigma_1 \dots \sigma_j), \end{aligned} \quad (\text{A.5})$$

where the coefficients $c_k^{[l, k-l]}$ satisfy

$$\sum_{j=0}^k c_j^{[l, k-l]} a^{k-j} b^j = (a - b)^l (a + b)^{k-l}. \quad (\text{A.6})$$

Now it is turn to evaluate the character $\chi_{(n_a, n_b)}$ for the element $\sigma_1 \dots \sigma_j$. The $O(k)$ character in (n_a, n_b) representation is given by

$$\chi_{(n_a, n_b)} = (h_{n_a} - h_{n_a-2})(h_{n_b} - h_{n_b-4}) - (h_{n_a+1} - h_{n_a-3})(h_{n_b-1} - h_{n_b-3}), \quad (\text{A.7})$$

where h_n is the homogeneous symmetric polynomials of the eigenvalues of the representation matrix. For the element $\Sigma_j = \sigma_1 \dots \sigma_j$, it is given by

$$\sum_{n=0}^{\infty} h_n(\Sigma_j) z^n = \frac{1}{\det(1 - z \Sigma_j)} = \frac{1}{(1 + z)^j (1 - z)^{k-j}}. \quad (\text{A.8})$$

Defining the multiplicities generating function

$$N^{[l, k-l]}(z, w) = \sum_{n_a=-1}^{\infty} \sum_{n_b=0}^{\infty} N_{(n_a, n_b)}^{[l, k-l]} z^{n_a} w^{n_b}, \quad (\text{A.9})$$

we obtain

$$N^{[l, k-l]}(z, w) = \frac{n(z, w)}{2^k} \sum_{j=0}^k \frac{c_j^{[l, k-l]}}{[(1 + z)(1 + w)]^j [(1 - z)(1 - w)]^{k-j}}. \quad (\text{A.10})$$

with

$$n(z, w) = (1 - z^2)(1 - w^4) - (z^{-1} - z^3)(w - w^3) = \frac{(1 - z^2)(1 - w^2)(z - w)(1 - zw)}{z}. \quad (\text{A.11})$$

Finally, using (A.6), we get a simple form,

$$N^{[l, k-l]}(z, w) = n(z, w) \left(\frac{z + w}{(1 - z^2)(1 - w^2)} \right)^l \left(\frac{1 + zw}{(1 - z^2)(1 - w^2)} \right)^{k-l}. \quad (\text{A.12})$$

As we shall see below, this generating function of multiplicities has a simple relation to the character.

Defining the generating function $N_{\mathcal{H}}(q, x)$ of the multiplicities as

$$N_{\mathcal{H}}(q, x) = \sum_{\Delta, s} N_{\mathcal{V}(\Delta, s)}^{\mathcal{H}} q^{\Delta} x^s, \quad (\text{A.13})$$

it has a simple relation to the one in (A.9) for $\mathcal{H} = \text{Di}^{\otimes l} \otimes \text{Rac}^{\otimes(k-l)}$ as

$$N_{\mathcal{H}}(q, x) = q^{\frac{k}{2}} N^{[l, k-l]} \left(\sqrt{q x}, \sqrt{q x^{-1}} \right). \quad (\text{A.14})$$

Let us first consider the cyclic tensor product of $\text{Di} \oplus \text{Rac}$.

$$\text{Cyc}^{\otimes p}(\text{Di} \oplus \text{Rac}) = \bigoplus_{n_a=0}^{\infty} \bigoplus_{n_b=0}^{n_a} N_{(n_a, n_b)}^{\text{cyc}(p)} D\left(\frac{n_a + n_b + p}{2}, \frac{n_a - n_b}{2}\right). \quad (\text{A.15})$$

Again, we consider the branching of $O(p)$ to its cyclic subgroup \mathbb{Z}_p :

$$\pi_{(n_a, n_b)}^{O(p)} \Big|_{\mathbb{Z}_p} = N_{(n_a, n_b)}^{\text{cyc}(p)} \pi_{\text{cyc}}^{\mathbb{Z}_p} \oplus \dots, \quad (\text{A.16})$$

where the multiplicity of the cyclic singlet can be obtained as

$$N_{(n_a, n_b)}^{\text{cyc}(p)} = \frac{1}{p} \sum_{l=0}^{p-1} \chi_{(n_a, n_b)}(C^l), \quad (\text{A.17})$$

where C is the cyclic permutation. The character with cyclic group element can be evaluated as

$$\sum_{n=0}^{\infty} h_n(C^l) z^n = \frac{1}{\det(1 - z C^l)} = \frac{1}{\left(1 - z^{\frac{p}{\gcd(l, p)}}\right)^{\gcd(l, p)}}. \quad (\text{A.18})$$

Finally, the multiplicities are generated by

$$\begin{aligned} N^{\text{cyc}(p)}(z, w) &= \sum_{n_a=-1}^{\infty} \sum_{n_b=0}^{\infty} N_{(n_a, n_b)}^{\text{cyc}(p)} z^{n_a} w^{n_b} \\ &= \frac{n(z, w)}{p} \sum_{l=0}^{p-1} \frac{1}{\left[\left(1 - z^{\frac{p}{\gcd(l, p)}}\right) \left(1 - w^{\frac{p}{\gcd(l, p)}}\right)\right]^{\gcd(l, p)}}. \end{aligned} \quad (\text{A.19})$$

B AdS₄ Zeta function

B.1 Modified Zeta function regularization

As we are only interested in the zeta function up to order z in the small z expansion, it is sufficient to check that the $\zeta_{\Delta, s}(z)$ and $\tilde{\zeta}_{\Delta, s}(z)$ agree up to order z^2 . To carry out this check, it is more convenient to work with $y = \cos \alpha$ instead of α itself. The character of $\mathcal{V}(\Delta, s)$ may be rewritten as

$$\chi_{\Delta, s}(\beta, y) = \frac{e^{-(\Delta - \frac{3}{2})\beta}}{4 \sinh \frac{\beta}{2}} \frac{V_{2s}(y)}{\cosh \beta - y}, \quad (\text{B.1})$$

where $V_n(y)$ is shorthand for $U_n(\sqrt{(y+1)/2})$, and U_n is the Chebyshev polynomial of the second kind. We need only first two Taylor coefficients of V_n .

$$V_n(1) = n + 1, \quad V'_n(1) = \frac{n(n+1)(n+2)}{12}. \quad (\text{B.2})$$

With these inputs, $\tilde{\zeta}_{\Delta,s}$ reduces to the following expressions:

$$\tilde{\zeta}_{\Delta,s}(z, 0) = \frac{2s+1}{6} \zeta_4\left(2z, \Delta - \frac{3}{2}\right) - \frac{s(s+1)(2s+1)}{6} \zeta_2\left(2z, \Delta - \frac{3}{2}\right) \quad (\text{B.3})$$

with

$$\zeta_n(z, a) = \frac{(n-1)!}{2^n} \int_0^\infty d\beta e^{-a\beta} \frac{\beta^{z-1}}{\Gamma(z)} \frac{\cosh \frac{\beta}{2}}{\sinh^n \frac{\beta}{2}}. \quad (\text{B.4})$$

The explicit evaluation of $\zeta_n(z, a)$ for $n = 2, 4$ yields

$$\begin{aligned} \zeta_2(z, a) &= \zeta\left(z-1, a + \frac{1}{2}\right) - a \zeta\left(z, a + \frac{1}{2}\right) \\ \zeta_4(z, a) &= \zeta\left(z-3, a + \frac{3}{2}\right) - 3a \zeta\left(z-2, a + \frac{3}{2}\right) \\ &\quad + \left(3a^2 - \frac{1}{4}\right) \zeta\left(z-1, a + \frac{3}{2}\right) - a \left(a^2 - \frac{1}{4}\right) \zeta\left(z, a + \frac{3}{2}\right). \end{aligned} \quad (\text{B.5})$$

From this, one can first calculate the constant part with $a = \Delta - \frac{3}{2}$,

$$\tilde{\zeta}_{a+\frac{3}{2},s}(0) = \frac{2s+1}{6} \left[\frac{a^4}{4} - \frac{a^2}{8} - \frac{17}{960} - s(s+1) \left(\frac{a^2}{2} + \frac{1}{24} \right) \right], \quad (\text{B.6})$$

using the identity $\zeta(-n, x) = -B_{n+1}(x)/(n+1)$, where $B_n(x)$ is Bernoulli polynomial. This result agrees with (4.3). Then we move to the first derivative part,

$$\begin{aligned} \tilde{\zeta}'_{a+\frac{3}{2},s}(0) &= \frac{2s+1}{3} \left[\zeta'\left(-3, a + \frac{3}{2}\right) - 3a \zeta'\left(-2, a + \frac{3}{2}\right) + \left(3a^2 - \frac{1}{4}\right) \zeta'\left(-1, a + \frac{3}{2}\right) \right. \\ &\quad \left. - a \left(a^2 - \frac{1}{4}\right) \zeta'\left(0, a + \frac{3}{2}\right) - s(s+1) \left[\zeta'\left(-1, a + \frac{1}{2}\right) - a \zeta'\left(0, a + \frac{1}{2}\right) \right] \right]. \end{aligned} \quad (\text{B.7})$$

In order to show the coincidence with (4.4), we first compare them for the value $a = 0$:

$$\begin{aligned} \tilde{\zeta}'_{\frac{3}{2},s}(0) &= \frac{2s+1}{3} \left[\zeta'\left(-3, \frac{1}{2}\right) - \left(s + \frac{1}{2}\right)^2 \zeta'\left(-1, \frac{1}{2}\right) \right], \\ \zeta'_{\frac{3}{2},s}(0) &= \frac{2s+1}{3} \left[c_3 + \left(s + \frac{1}{2}\right)^2 c_1 \right]. \end{aligned} \quad (\text{B.8})$$

One can check that $\zeta'(-3, \frac{1}{2}) = -c_3$ and $\zeta'(-1, \frac{1}{2}) = c_1$ so they match. Then, we consider their a derivatives, we get

$$\begin{aligned} \frac{\partial}{\partial a} \tilde{\zeta}'_{a+\frac{3}{2},s}(0) &= \frac{2s+1}{3} \left[-\zeta\left(-2, a+\frac{3}{2}\right) + 3a\zeta\left(-1, a+\frac{3}{2}\right) - \left(3a^2 - \frac{1}{4}\right)\zeta\left(0, a+\frac{3}{2}\right) \right. \\ &\quad \left. - a\left(a^2 - \frac{1}{4}\right)\psi\left(a+\frac{3}{2}\right) + s(s+1)a\left[\psi\left(a+\frac{1}{2}\right) + 1\right] \right] \\ &= \frac{2s+1}{3} \left[\frac{11}{6}a^3 - \frac{5}{24}a + s(s+1)a + \left[\left(s+\frac{1}{2}\right)^2 - a^2\right]a\psi\left(a+\frac{1}{2}\right) \right], \end{aligned} \quad (\text{B.9})$$

and

$$\frac{\partial}{\partial a} \zeta'_{a+\frac{3}{2},s}(0) = \frac{2s+1}{3} \left[\frac{1}{2}a^3 + \frac{1}{24}a + \left[\left(s+\frac{1}{2}\right)^2 - a^2\right]a\psi\left(a+\frac{1}{2}\right) \right], \quad (\text{B.10})$$

where we have used the following identities of Hurwitz zeta function,

$$\frac{\partial}{\partial a} \zeta(s, a) = -s \zeta(s+1, a), \quad \frac{\partial}{\partial a} \zeta'(0, a) = \psi(a). \quad (\text{B.11})$$

Finally, comparing (B.9) with (B.10), we get a non-trivial difference between two zeta functions:

$$\tilde{\zeta}_{a+\frac{3}{2},s}(z) - \zeta_{a+\frac{3}{2},s}(z) = \frac{1}{3} \left(s + \frac{1}{2}\right) a^2 \left[\frac{1}{6} a^2 - \left(s + \frac{1}{2}\right)^2 \right] z + \mathcal{O}(z^2). \quad (\text{B.12})$$

The above difference can be also expressed from the character as

$$\begin{aligned} \Delta \Gamma_{\mathcal{H}}^{(1)\text{ren}} &= \tilde{\Gamma}_{\mathcal{H}}^{(1)\text{ren}} - \Gamma_{\mathcal{H}}^{(1)\text{ren}} \\ &= \frac{1}{2\pi i} \oint d\beta \frac{2 \sinh^3 \frac{\beta}{2}}{\beta^3} \left(\frac{8}{3\beta^2} + \frac{2}{\sinh^2 \frac{\beta}{2}} - \frac{1}{3} + 4\partial_\alpha^2 \right) \chi_{\mathcal{H}}(\beta, \alpha) \Big|_{\alpha=0}, \end{aligned} \quad (\text{B.13})$$

where the contour encircles the $\beta = 0$ point. Hence, one can simply adjust the formula (4.24) by (B.13). One can notice that the difference term vanishes when the character $\chi_{\mathcal{H}}$ is even function of β .

B.2 Numerical approach to small β cut-off regularization

In section 4, we noted the difficulty of the analytic computation of $\tilde{\zeta}_{\log,k}(z)$. We can proceed numerically by change the regularization scheme from zeta function one to short β cut-off one:

$$\int_0^\infty d\beta \beta^{2z} f(\beta) \rightarrow \int_\epsilon^\infty d\beta f(\beta), \quad (\text{B.14})$$

where we simply remove all the divergence arising in the limit $\epsilon \rightarrow 0$. The absence of $\tilde{\zeta}_{\log,k}(0)$ can be verified in the new scheme from the absence of $1/\beta$ term of the integrand $f(\beta)$. Thanks to this, the splitting of $f(\beta)$ into the regular and singular parts is unambiguous. Truncating the integrand only to the regular part, the $\epsilon \rightarrow 0$ limit of the integral

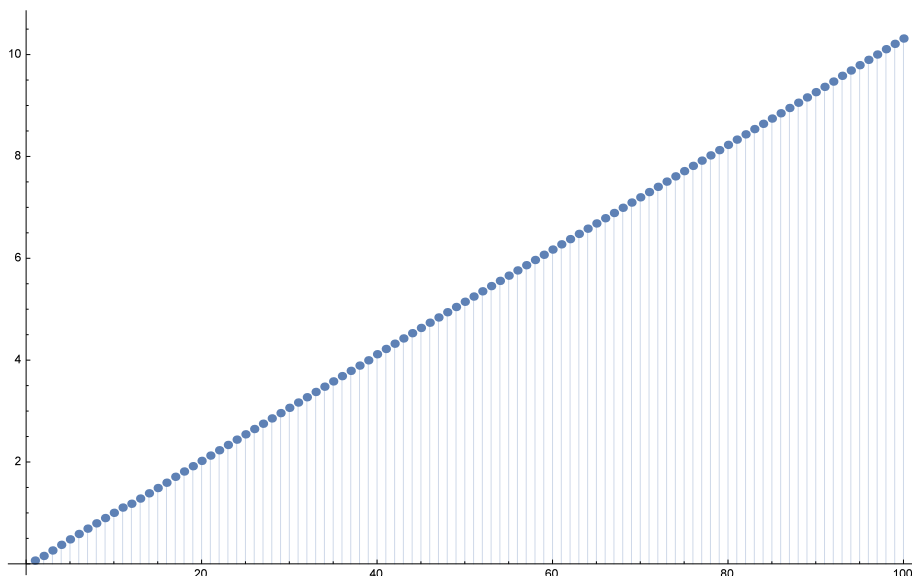


Figure 8. Vacuum energy of AdS_4 theory in different slices.

become well-defined and can be evaluated numerically. The plot in figure 8 shows the numerical values of the $\Gamma_{\log,k}^{(1)\text{ren}}$ contributions for $k = 1, 2, \dots, 100$. It shows an approximate linear growth with a very small perturbations:

$$\Gamma_{\log,k}^{(1)\text{ren}} \simeq -0.049 + 0.108 k, \quad (\text{B.15})$$

which a linear fit of our numerical data set. The total vacuum energy is given by

$$\Gamma_{\text{AdS}_4}^{(1)\text{ren}} = -\Gamma_{\text{Rac}}^{(1)\text{ren}} + \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \Gamma_{\log,k}^{(1)\text{ren}}. \quad (\text{B.16})$$

Since each of $\Gamma_{\log,k}^{(1)\text{ren}}$ is positive and growing almost linearly, the total vacuum energy is again given by a divergent series, to which we do not have an analytic access. We may nevertheless proceed the summation based on the numerical fitting (B.15) and the regularization of the Euler totient sum, $\sum_{k=1}^{\infty} \varphi(k) k^{-z} = \zeta(z-1)/\zeta(z)$ but the result²⁰ obtained in this way would be hardly reliable because of the unjustifiable combination of numerical approximations and analytic continuations.

²⁰If we plug in the approximation (B.15) into the above, we are lead to

$$\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} (-0.049 + 0.108 k) = -0.049 \frac{\zeta(0)}{\zeta(1)} + 0.108 \frac{\zeta(-1)}{\zeta(0)} \simeq 0.018. \quad (\text{B.17})$$

Combining with the contribution $-\Gamma_{\text{Rac}}^{(1)\text{ren}}$, we get the following value,

$$\Gamma_{\text{AdS}_4}^{(1)\text{ren}} \simeq 0.018 - 0.064 = -0.046. \quad (\text{B.18})$$

C AdS₅ zeta function

The zeta functions are given by β -integrals whose z -dependence regularizes the pole of the integrand at $\beta = 0$. Upon analytic continuations in z , the zeta functions give the vacuum energies by their first derivatives at $z = 0$. The typical way to evaluate these β -integrals is by partial fraction decomposition of the integrand in $q = e^{-\beta}$ and by using the integral representation of (derivatives of) Lerch zeta function,

$$\int_0^\infty d\beta \frac{\beta^{z-1} q^{n+a}}{(1-pq)^{n+1}} = \frac{\Gamma(z)}{n!} \partial_p^n \Phi(p, z, a). \quad (\text{C.1})$$

and the identity,

$$\Phi(p, z-1, a) = (a + p \partial_p) \Phi(p, z, a). \quad (\text{C.2})$$

We will illustrate this method very soon for the case of the Vasiliev theory in AdS₅. Also eventually for the matrix model we will need the function $\tilde{\Phi}$, defined in (5.56), given by

$$\tilde{\Phi}(p, z, a) = \int_0^\infty d\beta \frac{\beta^{z-1}}{\Gamma(z)} \log(1 - p e^{-\beta}) e^{-a\beta}, \quad (\text{C.3})$$

which is related to the Lerch zeta function via

$$\partial_p \tilde{\Phi}(p, z, a) = -\Phi(p, z, a+1). \quad (\text{C.4})$$

i.e. it is the p -primitive of Lerch function, up to shifts in the argument a . It is also useful to note that the function $\tilde{\Phi}$ has a series representation, given by

$$\tilde{\Phi}(p, z, a) = - \sum_{m=1}^{\infty} \frac{p^m}{m(m+a)^z}. \quad (\text{C.5})$$

C.1 Zeta function of the vector model

In this section we will present the detailed computations for the vacuum energy for the non-minimal and minimal Vasiliev theories in AdS₅, outlined respectively in (5.34) and (5.36). Let us consider first the zeta function for the non-minimal theory given by the first line of (5.34). We first rewrite the integrand as a rational function of $q = e^{-\beta}$, then do the partial fraction decomposition:

$$\begin{aligned} & \frac{\Gamma(z) \zeta_{\text{non-min}}(z)}{\log R} \\ &= \lim_{\epsilon \rightarrow 0} \int_0^\infty d\beta e^{-\epsilon\beta} \left[\frac{(1+q)^2 2^{1-2z} \beta^{2z-6}}{(1-q)^2 \Gamma(z-2)} + \frac{(1+q)^2 (1-10q+q^2) \beta^{2z-4}}{4^z \times 3(1-q)^4 \Gamma(z-1)} \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{2^{1-2z}}{\Gamma(z-2)} \int_0^\infty d\beta \beta^{2z-6} \left[q^\epsilon - 4 \frac{q^\epsilon}{1-q} + 4 \frac{q^\epsilon}{(1-q)^2} \right] \\ & \quad + \lim_{\epsilon \rightarrow 0} \frac{4^{-z}}{3\Gamma(z-1)} \int_0^\infty d\beta \beta^{2z-4} \left[q^\epsilon + 4 \frac{q^\epsilon}{1-q} - 36 \frac{q^\epsilon}{(1-q)^2} + 64 \frac{q^\epsilon}{(1-q)^3} - 32 \frac{q^\epsilon}{(1-q)^4} \right]. \end{aligned} \quad (\text{C.6})$$

Thus, *via* the partial fraction decomposition above, the β integrals have reduced integral representations for the Lerch zeta functions and their p derivatives, as in (C.1), evaluated at $p = 1$. We next use (C.2), and the identity $\Phi(1, z, a) = \zeta(z, a)$ to show that

$$\begin{aligned} \frac{\Gamma(z) \zeta_{\text{non-min}}(z)}{\log R} &= \frac{2^{3-2z} \Gamma(2z-5)}{\Gamma(z-2)} \left[-\zeta(2z-5, 0) + \zeta(2z-6, -1) + \zeta(2z-5, -1) \right] \\ &\quad + \frac{4^{1-z} \Gamma(2z-3)}{3\Gamma(z-1)} \left[-\frac{4}{3} \zeta(2z-6, -2) + 4\zeta(2z-5, -2) \right. \\ &\quad \left. + \frac{64}{3} \zeta(2z-4, -2) - 9\zeta(2z-4, -1) + 16\zeta(2z-3, -2) \right. \\ &\quad \left. - 9\zeta(2z-3, -1) + \zeta(2z-3, 0) \right] \\ &= \frac{\Gamma(z - \frac{5}{2}) \zeta(2z-6)}{8\sqrt{\pi}} - \frac{\Gamma(z - \frac{3}{2}) \zeta(2z-6)}{9\sqrt{\pi}} + \frac{\Gamma(z - \frac{3}{2}) \zeta(2z-4)}{36\sqrt{\pi}}, \quad (\text{C.7}) \end{aligned}$$

which is nothing but the last line of (5.34).

We now turn to the case of the minimal Vasiliev theory, in which case we need to evaluate the second line of (5.36). Following the method described above, we get

$$\begin{aligned} \frac{\Gamma(z) \zeta_R(z)}{\log R} &= \lim_{\epsilon \rightarrow 0} \int_0^\infty d\beta e^{-\epsilon\beta} \left[\frac{(1-q)(1+q^2)\beta^{2z-6}}{4^z(1+q)^3\Gamma(z-2)} + \frac{(1-q)(1-10q+q^2)(1+q^2)\beta^{2z-4}}{2^{2z+1}3(1+q)^5\Gamma(z-1)} \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{4^{-z}}{\Gamma(z-2)} \int_0^\infty d\beta \beta^{2z-6} \left[-q^\epsilon + \frac{4q^\epsilon}{1+q} - \frac{6q^\epsilon}{(1+q)^2} + \frac{4q^\epsilon}{(1+q)^3} \right] \\ &\quad + \frac{2^{-1-2z}}{3\Gamma(z-1)} \int_0^\infty d\beta \beta^{2z-4} \left[-q^\epsilon + \frac{16q^\epsilon}{1+q} - \frac{66q^\epsilon}{(1+q)^2} + \frac{124q^\epsilon}{(1+q)^3} - \frac{120q^\epsilon}{(1-q)^4} + \frac{48q^\epsilon}{(1-q)^5} \right] \\ &= -\frac{\Gamma(2z-5) [(2^{2z}-256)\zeta(2z-7) + (2^{2z}-64)\zeta(2z-5)]}{2^{4z-1}\Gamma(z-2)} \\ &\quad - \frac{\Gamma(2z-3)(2^{2z}-256)\zeta(2z-7)}{2^{4z} \times 3\Gamma(z-1)} \\ &\quad - \frac{2^{-4z}\Gamma(2z-3) [3(2^{1+2z}-128)\zeta(2z-5) + (2^{2z}-16)\zeta(2z-3)]}{3\Gamma(z-1)}. \quad (\text{C.8}) \end{aligned}$$

Again, using the the partial fraction expansion for the integrand, we could express all the β integrals in terms of the Lerch function and its derivatives, and eventually the Hurwitz zeta function. On further simplification, we obtained the third line of (5.36).

C.2 Zeta function for the matrix model

For the evaluation of the zeta function of the AdS theory dual to the free $\text{SU}(N)$ adjoint scalar CFT, we need to perform the integrals (5.21) with $f_{\log, k|n}$ as defined in (5.50) to arrive at the quantities $\zeta_{\log, k|n}$. We will show that these contributions to the zeta function vanish up to linear order in the small z expansion. Let us first focus on the integration of

the $f_{\log,k|2}$. The corresponding integral expression is given by

$$\begin{aligned} \frac{\Gamma(z) \zeta_{\log,k|2}(z)}{\log R} &= \int_0^\infty d\beta \frac{\left(\frac{\beta}{2}\right)^{2z-6}}{\Gamma(z-2)} \frac{\sinh^4 \frac{\beta}{2}}{2} \log \left[1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}} \right] \\ &= \int_0^\infty d\beta \frac{2^{1-2z} \beta^{2z-6}}{\Gamma(z-2)} \frac{(1-q)^4}{q^2} \log \left[\frac{1-4q^k+2q^{2k}-q^{3k}}{(1-q^k)^3} \right]. \end{aligned} \quad (\text{C.9})$$

Using eq. (5.57) we may write the log function as

$$\log \left[\frac{1-4q^k+2q^{2k}-q^{3k}}{(1-q^k)^3} \right] = \log \left[\frac{(1-p_1 q^k)(1-p_2 q^k)(1-p_3 q^k)}{(1-q^k)^3} \right], \quad (\text{C.10})$$

and express the zeta function as

$$\begin{aligned} \frac{\Gamma(z) \zeta_{\log,k|2}(z)}{\log R} &= \int_0^\infty d\beta \frac{2^{1-2z} \beta^{2z-6}}{\Gamma(z-2)} \frac{(1-q)^4}{q^2} \log \left[\frac{(1-p_1 q^k)(1-p_2 q^k)(1-p_3 q^k)}{(1-q^k)^3} \right] \\ &= \int_0^\infty d\beta \frac{2^{1-2z} \beta^{2z-6}}{\Gamma(z-2)} (q^{-2} - 4q^{-1} + 6 - 4q + q^2) \\ &\quad \times \log \left[\frac{(1-p_1 q^k)(1-p_2 q^k)(1-p_3 q^k)}{(1-q^k)^3} \right]. \end{aligned} \quad (\text{C.11})$$

On defining $\tilde{\beta} = k\beta$, and $\tilde{q} = e^{-\tilde{\beta}}$, we may rewrite the above expression as

$$\begin{aligned} \frac{\Gamma(z) \zeta_{\log,k|2}(z)}{\log R} &= \frac{(2k)^{5-2z}}{16 \Gamma(z-2)} \int_0^\infty d\tilde{\beta} \tilde{\beta}^{2z-6} \left(\tilde{q}^{-2/k} - 4\tilde{q}^{-1/k} + 6 - 4\tilde{q}^{1/k} + \tilde{q}^{2/k} \right) \\ &\quad \times \log \left[\frac{(1-p_1 \tilde{q})(1-p_2 \tilde{q})(1-p_3 \tilde{q})}{(1-\tilde{q})^3} \right]. \end{aligned} \quad (\text{C.12})$$

The above expression can be recast into (5.54) using (5.55) and (5.56), and evaluated using the contour integral (5.58).

While it is possible to evaluate the remaining contributions to the zeta function from $f_{\log,k|1}$ and $f_{\log,k|0}$ in this manner, we shall provide an additional means to do so. Consider the following series expansion of the logarithmic function,

$$\log \left[\frac{1-4q^k+2q^{2k}-q^{3k}}{(1-q^k)^3} \right] = - \sum_{\ell=1}^{\infty} \frac{1}{\ell} (p_1^\ell + p_2^\ell + p_3^\ell - 3) q^{k\ell}, \quad (\text{C.13})$$

where p_i 's are defined in (5.57). By plugging this series expression into the integral (C.9) and interchanging the order of ℓ summation and β integration, the latter can be evaluated to give

$$\begin{aligned} \frac{\Gamma(z) \zeta_{\log,k|2}(z)}{\log R} &= - \frac{(2k)^{5-2z} \Gamma(2z-5)}{16 \Gamma(z-2)} \sum_{\ell=1}^{\infty} \frac{1}{\ell} (p_1^\ell + p_2^\ell + p_3^\ell - 3) \times \\ &\quad \times \left[6 \ell^{5-2z} - 4 \left(\ell - \frac{1}{k} \right)^{5-2z} - 4 \left(\ell + \frac{1}{k} \right)^{5-2z} + \left(\ell - \frac{2}{k} \right)^{5-2z} + \left(\ell + \frac{2}{k} \right)^{5-2z} \right]. \end{aligned} \quad (\text{C.14})$$

We further simplify the expression to (5.54) by using the series form of $\tilde{\Phi}(p, z, a)$ given in (C.5) to write

$$\begin{aligned} & \sum_{\ell=1}^{\infty} \frac{1}{\ell} p_1^{\ell} \left[6\ell^{5-2z} - 4 \left(\ell - \frac{1}{k} \right)^{5-2z} - 4 \left(\ell + \frac{1}{k} \right)^{5-2z} + \left(\ell - \frac{2}{k} \right)^{5-2z} + \left(\ell + \frac{2}{k} \right)^{5-2z} \right] \\ &= 6 \tilde{\Phi}(p_1, 2z-5, 0) - 4 \tilde{\Phi}\left(p_1, 2z-5, \frac{1}{k}\right) - 4 \tilde{\Phi}\left(p_1, 2z-5, -\frac{1}{k}\right) \\ & \quad + \tilde{\Phi}\left(p_1, 2z-5, \frac{2}{k}\right) + \tilde{\Phi}\left(p_1, 2z-5, -\frac{2}{k}\right) = \Psi_k(p_1, z), \end{aligned} \quad (\text{C.15})$$

where Ψ_k was defined in (5.55). Hence we recover (5.54). Note however that the series (C.15) is not convergent since one of $|p_i|$ is greater than 1. Nevertheless, once the series is evaluated in its domain of convergence, it may be extended to $|p_i| > 1$ as well through its representation in terms of $\tilde{\Phi}$. The small z expansion of this expression has already been carried out in the main text and it was shown that the expression vanishes up to linear order in z . That is,

$$\zeta'_{\log, k|2}(0) = 0. \quad (\text{C.16})$$

In fact, there exists a shorter way to draw the same conclusion relying on an ad-hoc regularization prescription. Since we are interested in the zeta function up to linear order in z , it is easy to see that

$$\begin{aligned} \frac{\zeta_{\log, k|2}(z)}{\log R} &= -z \frac{k^5}{60} [\Psi(p_1, 0) + \Psi(p_2, 0) + \Psi(p_3, 0) - 3\Psi(1, 0)] + \mathcal{O}(z^2) \\ &= -z \frac{k^5}{60} \sum_{\ell=1}^{\infty} \left(p_1^{\ell} + p_2^{\ell} + p_3^{\ell} - 3 \right) + \mathcal{O}(z^2) \\ &= -z \frac{k^5}{60} \left(\sum_{i=1}^3 \frac{p_i}{1-p_i} - \frac{3}{2} \right) + \mathcal{O}(z^2) = \mathcal{O}(z^2). \end{aligned} \quad (\text{C.17})$$

In the above, the z independent part in the summand gives a divergent series for some roots p_i and for the -3 term. We have used the regularization prescriptions,

$$\sum_{\ell=1}^{\infty} p_i^{\ell} \rightarrow \frac{p_i}{1-p_i}, \quad \sum_{\ell=1}^{\infty} 1 \rightarrow \zeta(0) = -\frac{1}{2}, \quad (\text{C.18})$$

to evaluate those sums, and subsequently also used the identity (5.57) to show that the order z term vanishes, thus obtaining the same result.

For the sake of simplicity, we shall use the above prescription to compute $\zeta'_{\log, k|1}(0)$ and $\zeta'_{\log, k|0}(0)$ in the rest of this section. We begin with the evaluation of $\zeta_{\log, k|1}$:

$$\frac{\Gamma(z) \zeta_{\log, k|1}(z)}{\log R} = I_1(z) + I_2(z), \quad (\text{C.19})$$

with

$$I_1(z) = \int_0^\infty d\beta \frac{\left(\frac{\beta}{2}\right)^{2z-4}}{\Gamma(z-1)} \frac{\sinh^2 \frac{\beta}{2} \left(\sinh^2 \frac{\beta}{2} - 3\right)}{6} \log \left(1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}}\right), \quad (\text{C.20})$$

$$I_2(z) = \int_0^\infty d\beta \frac{\left(\frac{\beta}{2}\right)^{2z-4}}{\Gamma(z-1)} \frac{k^2 \sinh^4 \frac{\beta}{2} \coth \frac{k\beta}{2}}{\sinh(k\beta) - 8 \sinh^4 \frac{k\beta}{2}}. \quad (\text{C.21})$$

Rewriting the integrands in terms of q , they become

$$I_1(z) = \int_0^\infty d\beta \frac{2^{-2z-1} \beta^{2z-4}}{3\Gamma(z-1)} \frac{(1-q)^2 (1-14q+q^2)}{q^2} \log \left(\frac{1-4q^k+2q^{2k}-q^{3k}}{(1-q^k)^3}\right), \quad (\text{C.22})$$

$$I_2(z) = - \int_0^\infty d\beta \frac{k^2 2^{1-2z} \beta^{2z-4}}{\Gamma(z-1)} \frac{(1-q)^4 q^{2k-2} (1+q^k)}{(1-q^k)^2 (1-4q^k+2q^{2k}-q^{3k})}. \quad (\text{C.23})$$

Next, applying the series expansion (C.13), I_1 may be expressed as

$$\begin{aligned} I_1(z) &= - \frac{(2k)^{3-2z} \Gamma(2z-3)}{48 \Gamma(z-1)} \sum_{\ell=1}^{\infty} \frac{p_1^\ell + p_2^\ell + p_3^\ell - 3}{\ell} \times \\ &\quad \times \left[30 \ell^{3-2z} - 16 \left(\ell - \frac{1}{k}\right)^{3-2z} - 16 \left(\ell + \frac{1}{k}\right)^{3-2z} + \left(\ell - \frac{2}{k}\right)^{3-2z} + \left(\ell + \frac{2}{k}\right)^{3-2z} \right] \\ &= - \frac{(2k)^{3-2z} \Gamma(2z-3)}{48 \Gamma(z-1)} [\Psi^{(1)}(p_1, z) + \Psi^{(1)}(p_2, z) + \Psi^{(1)}(p_3, z) - 3\Psi^{(1)}(1, z)], \end{aligned} \quad (\text{C.24})$$

where the function $\Psi^{(1)}(p, z)$ is defined by

$$\begin{aligned} \Psi^{(1)}(p, z) &= -30 \tilde{\Phi}(p_1, 2z-3, 0) - 16 \tilde{\Phi}\left(p_1, 2z-3, -\frac{1}{k}\right) - 16 \tilde{\Phi}\left(p_1, 2z-3, \frac{1}{k}\right) \\ &\quad + \tilde{\Phi}\left(p_1, 2z-3, -\frac{2}{k}\right) + \tilde{\Phi}\left(p_1, 2z-3, \frac{2}{k}\right). \end{aligned} \quad (\text{C.25})$$

The leading behavior of $I_1(z)$ is given by

$$\begin{aligned} I_1(z) &= \frac{k^3}{36} [\Psi^{(1)}(p_1, 0) + \Psi^{(1)}(p_2, 0) + \Psi^{(1)}(p_3, 0) - 3\Psi^{(1)}(1, 0)] \\ &= \sum_{\ell=1}^{\infty} k [p_1^\ell + p_2^\ell + p_3^\ell - 3] + \mathcal{O}(z). \end{aligned} \quad (\text{C.26})$$

Therefore, following the prescription (C.18), we obtain

$$I_1(0) = k \left(\frac{p_1}{1-p_1} + \frac{p_2}{1-p_2} + \frac{p_3}{1-p_3} + \frac{3}{2} \right) = 0. \quad (\text{C.27})$$

For the integrand of $I_2(z)$, we have the series expansion,

$$\frac{q^{2k} (1+q^k)}{(1-q^k)^2 (1-4q^k+2q^{2k}-q^{3k})} = \sum_{\ell=0}^{\infty} \left(f_1 p_1^\ell + f_3 p_2^\ell + f_2 p_3^\ell - \ell \right) q^{k\ell}, \quad (\text{C.28})$$

where p_i 's are defined in the (5.57) and f_i 's are three roots of the cubic equation $107f^3 + 3f - 2 = 0$. Applying this expansion, (C.23) becomes

$$\begin{aligned}
 I_2(z) &= -\frac{(2k)^{5-2z} \Gamma(2z-3)}{16 \Gamma(z-1)} \sum_{\ell=0}^{\infty} \left(f_1 p_1^\ell + f_2 p_2^\ell + f_3 p_3^\ell + \ell \right) \times \\
 &\quad \times \left(6 \ell^{3-2z} - 4 \left(\ell - \frac{1}{k} \right)^{3-2z} - 4 \left(\ell + \frac{1}{k} \right)^{3-2z} + \left(\ell - \frac{2}{k} \right)^{3-2z} + \left(\ell + \frac{2}{k} \right)^{3-2z} \right) \\
 &= -\frac{(2k)^{5-2z} \Gamma(2z-3)}{16 \Gamma(z-1)} \left[\Xi^{(1)}(p_1, 0) + \Xi^{(1)}(p_2, 0) + \Xi^{(1)}(p_3, 0) + 6 \zeta(2z-4) \right. \\
 &\quad \left. - 3 \Phi^+ \left(1, 2z-4, \frac{1}{k} \right) + \frac{3}{k} \Phi^- \left(1, 2z-3, \frac{1}{k} \right) \right], \quad (C.29)
 \end{aligned}$$

where Φ^\pm is defined by

$$\Phi^\pm(p, z, a) = \Phi(p, z, a) \pm \Phi(p, z, -a), \quad (C.30)$$

and $\Xi^{(1)}$ by

$$\Xi^{(1)}(p, z) = 6 \Phi(p, 2z-3, 0) - 4 \Phi^+ \left(p, 2z-3, \frac{1}{k} \right) + \Phi^+ \left(p, 2z-3, \frac{2}{k} \right). \quad (C.31)$$

Using an analogous prescription of (C.18), we get $I_2(0) = 0$ at leading order of z . Hence, we conclude that $\zeta'_{\log, k|1}(0) = 0$.

Finally, we consider the integral of $f_{\log, k|0}$ given by

$$\frac{\Gamma(z) \zeta_{\log, k|0}(z)}{\log R} = J_1(z) + J_2(z) + J_3(z), \quad (C.32)$$

with

$$J_1(z) = -\frac{2}{3} \int_0^\infty d\beta \frac{\left(\frac{\beta}{2} \right)^{2z-2}}{\Gamma(z)} k^2 \left[(k^2 - 1) \sinh^2 \frac{\beta}{2} + 3 \right] \frac{\sinh^2 \frac{\beta}{2} \coth \frac{k\beta}{2}}{\sinh(k\beta) - 8 \sinh^4 \frac{k\beta}{2}}, \quad (C.33)$$

$$J_2(z) = -2 \int_0^\infty d\beta \frac{\left(\frac{\beta}{2} \right)^{2z-2}}{\Gamma(z)} k^4 \left(\frac{\sinh^2 \frac{\beta}{2} \coth \frac{k\beta}{2}}{\sinh(k\beta) - 8 \sinh^4 \frac{k\beta}{2}} \right)^2, \quad (C.34)$$

$$J_3(z) = \int_0^\infty d\beta \frac{\left(\frac{\beta}{2} \right)^{2z-2}}{\Gamma(z)} \log \left[1 - \frac{\sinh(k\beta)}{8 \sinh^4 \frac{k\beta}{2}} \right]. \quad (C.35)$$

The integrands of the above can be expressed in terms of q as

$$\begin{aligned}
 J_1(z) &= \int_0^\infty d\beta \frac{k^2 4^{-z} \beta^{2z-2}}{\Gamma(z)} \frac{(1-q)^2 [(k^2-1)q^2 - (2k^2-14)q + k^2-1] q^{2k-2} (1+q^k)}{3(1-q^k)^2 (1-4q^k + 2q^{2k} - q^{3k})}, \\
 J_2(z) &= -\int_0^\infty d\beta \frac{k^4 2^{1-2z} \beta^{2z-2}}{\Gamma(z)} \frac{(1-q)^4 q^{4k-2} (1+q^k)^2}{(1-q^k)^4 (1-4q^k + 2q^{2k} - q^{3k})^2}, \\
 J_3(z) &= \int_0^\infty d\beta \frac{4^{1-z} \beta^{2z-2}}{\Gamma(z)} \log \left[\frac{1-4q^k + 2q^{2k} - q^{3k}}{(1-q^k)^3} \right]. \quad (C.36)
 \end{aligned}$$

For the integrand of J_1 and J_3 , we use again the series expansion (C.13) and (C.28), respectively. In case of J_2 , we have the expansion,

$$\frac{q^{4k}(1+q^k)^2}{(1-q^k)^4(1-4q^k+2q^{2k}-q^{2k})^2} = \sum_{l=0}^{\infty} \left[l \left(c_1 p_1^{\ell+2} + e_2 p_2^{\ell+1} + e_3 p_3^{\ell+1} \right) + d_1 p_1^{\ell} + d_3 p_2^{\ell} + d_2 p_3^{\ell} + \frac{\ell^3 - \ell + 1}{6} \right] q^{k\ell}. \quad (\text{C.37})$$

Here the constant c_1 is the real solution of the cubic equation $11449x^3 - 6527x^2 + 957x - 4 = 0$. The e_2 and e_3 are imaginary solutions of the cubic equation $11449x^3 - 1391x^2 + 283x - 4 = 0$. And, the constants d_1, d_2 and d_3 are the three solutions of the cubic equation $1225043x^3 + 1225043x^2 + 409177x + 45401 = 0$. Applying above series expansions, the function J_1 is given by

$$\begin{aligned} J_1(z) &= - \sum_{\ell=1}^{\infty} \int_0^{\infty} d\beta \frac{(k^2(1-e^{-\beta})^2 [(k^2-1)e^{-2\beta} - (2k^2-14)e^{-\beta} + k^2-1] 4^{-z}\beta^{2z-2})}{3\Gamma(z)} \\ &\quad \times \left(f_1 p_1^{\ell} + f_2 p_2^{\ell} + f_3 p_3^{\ell} + \ell \right) e^{2\beta} e^{-k\beta\ell} \\ &= - \frac{4^{-z} k^{3-2z} \Gamma(2z-1)}{3\Gamma(z)} \left[f_1 \Xi^{(2)}(p_1, z) + f_2 \Xi^{(2)}(p_2, z) + f_3 \Xi^{(2)}(p_3, z) \right. \\ &\quad + 6(k^2-5)\zeta(2z-2) + (k^2-1)\Phi^+\left(1, 2z-2, \frac{2}{k}\right) \\ &\quad - 4(k^2-4)\Phi^+\left(1, 2z-2, \frac{1}{k}\right) + 4\left(k-\frac{4}{k}\right)\Phi^-\left(1, 2z-1, \frac{1}{k}\right) \\ &\quad \left. - 2\left(k-\frac{1}{k}\right)\Phi^-\left(1, 2z-1, \frac{2}{k}\right) \right]. \quad (\text{C.38}) \end{aligned}$$

Here the definition of $\Xi^{(2)}$ is

$$\begin{aligned} \Xi^{(2)}(p, z) &= 6(k^2-5)\Phi(p, 2z-1, 0) + (k^2-1)\Phi^+\left(p, 2z-1, \frac{2}{k}\right) \\ &\quad + (16-4k^2)\Phi^+\left(p, 2z-1, \frac{1}{k}\right). \quad (\text{C.39}) \end{aligned}$$

For the function J_2 , we get

$$\begin{aligned} J_2(z) &= - \sum_{\ell=1}^{\infty} \int_0^{\infty} d\beta \frac{k^4 (1-e^{-\beta})^4 2^{1-2z} \beta^{2z-2}}{\Gamma(z)} \\ &\quad \times \left(l \left(c_1 p_1^{\ell+2} + e_2 p_2^{\ell+1} + e_3 p_3^{\ell+1} \right) + d_1 p_1^{\ell} + d_3 p_2^{\ell} + d_2 p_3^{\ell} + \frac{\ell^3 - \ell + 1}{6} \right) e^{2\beta} e^{-k\beta\ell} \\ &= \frac{\Gamma(2z-1) 4^{-z} k^{5-2z}}{3\Gamma(z)} \left[d_1 \Xi^{(3)}(p_1, z) + d_3 \Xi^{(3)}(p_2, z) + d_2 \Xi^{(3)}(p_3, z) \right. \\ &\quad + c_1 p_1^2 \Xi^{(4)}(p_1, z) + e_2 p_2 \Xi^{(4)}(p_2, z) + e_3 p_3 \Xi^{(4)}(p_3, z) + 6\zeta(2z-1) \\ &\quad \left. - 6\zeta(2z-2) + 6\zeta(2z-4) - 4\Phi^+\left(1, 2z-4, \frac{1}{k}\right) + \Phi^+\left(1, 2z-4, \frac{2}{k}\right) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{12}{k} \Phi^- \left(1, 2z - 3, \frac{1}{k} \right) - \frac{6}{k} \Phi^- \left(1, 2z - 3, \frac{2}{k} \right) - 4 \left(1 + \frac{3}{k^2} \right) \Phi^+ \left(1, 2z - 2, \frac{1}{k} \right) \\
& + \left(1 + \frac{12}{k^2} \right) \Phi^+ \left(1, 2z - 2, \frac{2}{k} \right) - 4 \left(1 + \frac{1}{k} + \frac{1}{k^3} \right) \Phi^+ \left(1, 2z - 1, \frac{1}{k} \right) \\
& + \left(1 - \frac{2}{k} - \frac{8}{k^3} \right) \Phi^+ \left(1, 2z - 1, \frac{2}{k} \right) + \frac{4}{k} \Phi^- \left(1, 2z - 1, \frac{1}{k} \right) - \frac{2}{k} \Phi^+ \left(1, 2z - 1, \frac{2}{k} \right) \\
& + \frac{4}{k^3} \Phi^- \left(1, 2z - 1, \frac{1}{k} \right) - \frac{8}{k^3} \Phi^+ \left(1, 2z - 1, \frac{2}{k} \right) \Big]. \tag{C.40}
\end{aligned}$$

In this expression, $\Xi^{(3)}(p, z)$ and $\Xi^{(4)}(p, z)$ are defined by

$$\begin{aligned}
\Xi^{(3)}(p, z) &= 36 \Phi(p, 2z - 1, 0) + 6 \Phi^+ \left(p, 2z - 1, \frac{2}{k} \right) - 24 \Phi^+ \left(p, 2z - 1, \frac{1}{k} \right), \\
\Xi^{(4)}(p, z) &= \frac{6}{k p} \left[6 k p \Phi(p, 2z - 2, 0) + k \Phi^+ \left(p, 2z - 2, \frac{2}{k} \right) - 4 k \Phi^+ \left(p, 2z - 2, \frac{1}{k} \right) \right. \\
&\quad \left. - 2 \Phi^- \left(p, 2z - 1, \frac{2}{k} \right) + 4 \Phi^- \left(p, 2z - 1, \frac{1}{k} \right) \right]. \tag{C.41}
\end{aligned}$$

Using the prescription (C.18), it is straightforward to check that $J_1(z) + J_2(z)$ vanishes at leading order in z . Finally, for the function J_3 , we get

$$\begin{aligned}
J_3(z) &= - \sum_{\ell=1}^{\infty} \int_0^{\infty} d\beta \frac{4^{1-z} \beta^{2z-2}}{\Gamma(z)} \frac{(p_1^{\ell} + p_2^{\ell} + p_3^{\ell} - 3)}{\ell} e^{-k \beta \ell} \\
&= - \frac{(4^{1-z} \beta^{2z-2} k^{1-2z})}{\Gamma(z)} \left[\Phi(p_1, 1 - 2z, 0) + \Phi(p_2, 1 - 2z, 0) \right. \\
&\quad \left. + \Phi(p_3, 1 - 2z, 0) - 3 \Phi(1, 1 - 2z, 0) \right]. \tag{C.42}
\end{aligned}$$

Again through the prescription of (C.18), we get

$$\begin{aligned}
J_3(z) &= \sum_{\ell=1}^{\infty} 2k \left(p_1^{\ell} + p_2^{\ell} + p_3^{\ell} - 3 \right) + \mathcal{O}(z) \\
&= - \left(\frac{2 p_1}{p_1 - 1} + \frac{2 p_2}{p_2 - 1} + \frac{2 p_3}{p_3 - 1} - 3 \right) k + \mathcal{O}(z) = \mathcal{O}(z). \tag{C.43}
\end{aligned}$$

This completes the computation of the zeta function up to quadratic order in z within Prescription 2. We find that the linear term in z is absent hence the corresponding vacuum energy vanishes.

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